ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 255 - Vector Calculus and Linear Algebra

FINAL EXAMINATION

24.05.2017

SAMPLE SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTORS: **EMT**

DURATION: 120 minutes

Question	Grade	Out of
1		20
2		18
3		10
4		16
5		24
6		12
Total		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1) Let S be the part of the paraboloid $z = 6 - x^2 - y^2$ that lies above the plane $z = 2$, oriented upward. Let $\vec{F} = x^2y\vec{i} + y^2z\vec{j} + z^3\vec{k}$.

a) Evaluate $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet \vec{n}dS$, directly. $f(x,y) = 6 - x^2 - y^2$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^3 \end{vmatrix} = \left(\frac{\partial z^3}{\partial y} - \frac{\partial y^2z}{\partial z}\right)\vec{i} - \left(\frac{\partial z^3}{\partial x} - \frac{\partial x^2y}{\partial z}\right)\vec{j} + \left(\frac{\partial y^2z}{\partial x} - \frac{\partial x^2y}{\partial y}\right)\vec{k}$$

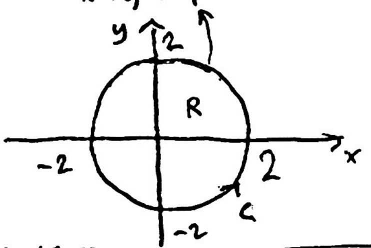
$$= (-y^2, 0, -x^2)$$

$$\vec{n}dS = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{(-f_x)^2 + (-f_y)^2 + 1}} \cdot \sqrt{(-f_x)^2 + (-f_y)^2 + 1}\, dxdy = \langle -(-2x), -(-2y), 1\rangle\, dxdy$$
$$= \langle 2x, 2y, 1\rangle\, dx\, dy.$$

Intersection of $z = 6 - x^2 - y^2$ & $z = 2$

$6 - x^2 - y^2 = 2$

$x^2 + y^2 = 4$



$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 2$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}dS = \iint_R \langle -y^2, 0, -x^2\rangle \cdot \langle 2x, 2y, 1\rangle\, dx\, dy$$
$$= \iint_R (-2xy^2 - x^2)\, dx\, dy = \int_0^{2\pi}\int_0^2 (-2r\cos\theta\, r^2\sin^2\theta - r^2\cos^2\theta)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^2 -2r^4\sin^2\theta\cos\theta\, dr\, d\theta + \int_0^{2\pi}\int_0^2 -r^3\cos^2\theta\, dr\, d\theta$$
$$= \int_0^{2\pi} -\frac{2}{5}r^5\sin^2\theta\cos\theta\,\Big|_0^2 d\theta + \int_0^{2\pi} -\frac{r^4}{4}\,\frac{1+\cos 2\theta}{2}\,\Big|_0^2 d\theta$$
$$= \int_0^{2\pi} -\frac{64}{5}\sin^2\theta\cos\theta\, d\theta - \int_0^{2\pi}(2 + 2\cos 2\theta)\, d\theta$$
$$= -\frac{64}{15}\sin^3\theta\,\Big|_0^{2\pi} - 2\theta - \sin 2\theta\,\Big|_0^{2\pi}$$
$$= -\frac{64}{15}\sin^3 2\pi + \frac{64}{15}\sin^3 0 - 2\cdot 2\pi + 2\cdot 0 - \sin 4\pi + \sin 0$$
$$= -4\pi$$

b) Evaluate $\oint_C \vec{F} \bullet d\vec{r}$, directly.

$\vec{r}(t) = \langle 2\cos t, 2\sin t, 2\rangle \qquad 0 \leq t \leq 2\pi.$

$d\vec{r} = \langle -2\sin t, 2\cos t, 0\rangle\, dt$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle x^2y, y^2z, z^3\rangle \cdot \langle dx, dy, dz\rangle$$
$$= \int_0^{2\pi} \langle 4\cos^2 t \cdot 2\sin t,\ 4\sin^2 t \cdot 2,\ 8\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\, dt$$
$$= \int_0^{2\pi} (-16\sin^2 t\cos^2 t + 16\sin^2 t\cos t)\, dt = \int_0^{2\pi}(-4\sin^2 2t)\, dt + 16\int_0^{2\pi}\sin^2 t\cos t\, dt$$
$$= \int_0^{2\pi} -4\cdot\frac{1-\cos 4t}{2}\, dt + 16\,\frac{\sin^3 t}{3}\,\Big|_0^{2\pi} = -2t - \frac{2}{4}\sin 4t\,\Big|_0^{2\pi} + \frac{16}{3}\sin^3 2\pi - \frac{16}{3}\sin^3 0$$
$$= -2\cdot 2\pi - \frac{1}{2}\sin 8\pi + 2\cdot 0 + \frac{1}{2}\sin 0 = -4\pi$$

c) Verify the Stoke's Theorem.

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}dS = -4\pi = \oint_C \vec{F} \cdot d\vec{r}.$$

2) Let $W = \{(x, y, z, t) \in \mathbb{R}^4 \,|\, x - 2y = 0,\, z + t = 0\}$.

(a) Show that W is a subspace of $\mathbb{R}^4$.

- $0 - 2.0 = 0$ & $0 + 0 = 0$. So $(0,0,0,0) \in W$. Hence, $W \neq \emptyset$.
- Let $u, w \in W$. Then $u = (x, y, z, t) \in \mathbb{R}^4$ with $x - 2y = 0$ and $z + t = 0$

and $w = (a, b, c, d)$ with $a - 2b = 0$ and $c + d = 0$.

$u + w = (x+a, y+b, z+c, t+d) \in \mathbb{R}^4$ and

$x + a - 2(y+b) = x + a - 2y - 2b = x - 2y + a - 2b = 0 + 0 = 0$

$z + c + t + d = z + t + c + d = 0 + 0 = 0$. So $u + w \in W$.

- Let $c \in \mathbb{R}$. $cu = (cx, cy, cz, ct) \in \mathbb{R}^4$ and

$cx - 2cy = c(x - 2y) = c.0 = 0$

$cz + ct = c(z+t) = c.0 = 0$ So $cu \in W$. Thus W is a subspace of $\mathbb{R}^4$.

(b) Find a basis S for W.

Let $u \in W$. Then $u = (x, y, z, t) \in \mathbb{R}^4$ with $x - 2y = 0$ and $z + t = 0$.

Then $x = 2y$ and $z = -t$.

So $u = (2y, y, -t, t) = (2y, y, 0, 0) + (0, 0, -t, t) = (2,1,0,0)y + (0,0,-1,1)t$

So $\{(2,1,0,0), (0,0,-1,1)\}$ is a spanning set for W.

$(2,1,0,0)y + (0,0,-1,1)t = (0,0,0,0) \rightarrow (2y, y, -t, t) = (0,0,0,0)$

So $y = t = 0$

Thus $\{(2,1,0,0), (0,0,-1,1)\}$ is a linearly independent set of vectors.

Hence $\{(2,1,0,0), (0,0,-1,1)\}$ is a basis for W.

$S =$

(c) Find a basis B for $\mathbb{R}^4$ containing S.

$I = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is the standard basis for $\mathbb{R}^4$. So $S \cup I$ is a spanning set for $\mathbb{R}^4$.

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 + R_3]{-2R_2 + R_1} \begin{bmatrix} 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \leftrightarrow R_4]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{cccccc} \uparrow & \uparrow & \uparrow & & \uparrow & \end{array} \end{array}$$

So $\{(2,1,0,0), (0,0,-1,1), (1,0,0,0), (0,0,1,0)\}$ is a basis for $\mathbb{R}^4$ containing S.

3) For each $u = (u_1, u_2)$, $w = (w_1, w_2) \in \mathbb{R}^2$, define $(u,\, w) = u_1w_1 - 2u_1w_2 - 2u_2w_1 + 5u_2w_2$.

a) Determine whether $(u,\, w)$ is an inner product or not. Let $u=(u_1,u_2)$, $v=(v_1,v_2)$, $w=(w_1,w_2)$, $c \in \mathbb{R}$.

- $(u,u) = u_1u_1 - 2u_1u_2 - 2u_2u_1 + 5u_2u_2$

$= u_1^2 - 4u_1u_2 + 5u_2^2 = u_1^2 - 4u_1u_2 + 4u_2^2 + u_2^2 = (u_1-2u_2)^2 + u_2^2 \geq 0$

$(u,u)=0 \iff (u_1-2u_2)^2+u_2^2=0 \iff u_1-2u_2=0$ & $u_2=0$

$\iff u_1=0$ & $u_2=0$

- $(w,u) = w_1u_1 - 2w_1u_2 - 2w_2u_1 + 5w_2u_2$

$= u_1w_1 - 2u_2w_1 - 2u_1w_2 + 5u_2w_2 = u_1w_1 - 2u_1w_2 - 2u_2w_1 + 5u_2w_2 = (u,w)$

- $(u+v, w) = ((u_1+v_1, u_2+v_2), (w_1,w_2))$

$= (u_1+v_1)w_1 - 2(u_1+v_1)w_2 - 2(u_2+v_2)w_1 + 5(u_2+v_2)w_2$

$= u_1w_1 + v_1w_1 - 2u_1w_2 - 2v_1w_2 - 2u_2w_1 - 2v_2w_1 + 5u_2w_2 + 5v_2w_2$

$= u_1w_1 - 2u_1w_2 - 2u_2w_1 + 5u_2w_2 + v_1w_1 - 2v_1w_2 - 2v_2w_1 + 5v_2w_2$

$= (u,w) + (v,w)$

- $(cu,w) = ((cu_1,cu_2),(w_1,w_2)) = cu_1w_1 - 2cu_1w_2 - 2cu_2w_1 + 5cu_2w_2$

$= c(u_1w_1 - 2u_1w_2 - 2u_2w_1 + 5u_2w_2) = c(u,w).$

Thus (u,w) is an inner product

b) For $u = (1, -2)$ and $w = (1, 1)$ evaluate (u, w).

$(u,w) = 1.1 - 2.1.1 - 2.(-2).1 + 5.(-2).1$

$= 1 - 2 + 4 - 10 = -7$

4) Let $V = \mathbb{R}^3$, $v_1 = (1,0,1)$, $v_2 = (0,1,1)$, $v_3 = (1,1,0)$, $w_1 = (1,0,0)$, $w_2 = (1,1,0)$, $w_3 = (1,1,1)$.

a) Show that $S = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

Since $\dim \mathbb{R}^3 = 3$ and S contains 3 elements it suffices to show S is linearly independent.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3+R_2]{+\frac{1}{2}R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are 3 leading entries in the reduced row echelon form for 3 vectors, S is linearly independent.
Therefore, S is a basis for $\mathbb{R}^3$.

b) It is given that $T = \{w_1, w_2, w_3\}$ is a basis for $\mathbb{R}^3$. Find the transition matrix $P_{S \leftarrow T}$ from the basis T to the basis S.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & -1 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{1}{2} \end{array}\right]$$

So $P_{S \leftarrow T} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & 1 & 1/2 \end{bmatrix}$

$$\xrightarrow[-R_3+R_2]{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1 & 1/2 \end{array}\right]$$

c) Compute the coordinate vector $[\alpha]_T$ of $\alpha = (2, -1, 3)$ with respect to T.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So $[\alpha]_T = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix}$

d) Compute the coordinate vector $[\alpha]_S$ of $\alpha = (2, -1, 3)$ with respect to S.

$$[\alpha]_S = P_{S \leftarrow T} [\alpha]_T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{3}{2} \\ -\frac{3}{2} + \frac{3}{2} \\ \frac{3}{2} - 4 + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

5) Let $A = \begin{bmatrix} 1 & -1 & -3 & 2 & 1 \\ 1 & 0 & -2 & 2 & -1 \\ -1 & 0 & 2 & -1 & 0 \end{bmatrix}$ be a 3×5 matrix whose reduced row echelon form is $R = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$.

(a) Find a basis for the row space of A, if possible.

$\{[1\ 0\ -2\ 0\ 1], [0\ 1\ 1\ 0\ -2], [0\ 0\ 0\ 1\ -1]\}$

(b) Find a basis for the column space of A, if possible.

$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$

(c) Find a basis β for the solution space of the system $Ax = 0$.

$x - 2z + u = 0$
$y + z - 2u = 0$
$t - u = 0$

$\longrightarrow$

$x = 2z - u$
$y = -z + 2u$
$t = u$

$\begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} z + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} u$.

$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) Find the rank and nullity of A.

rank $A = 3$

nullity $A = 2$

(e) Find the vector α, where $[\alpha]_\beta = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\alpha = 1 \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ -1-2 \\ 1-0 \\ 0-1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

(f) Determine whether or not $\gamma = (0, 2, 0, 3, -1)$ is a solution of the system $Ax = 0$.

$0 - 2 - 3 \cdot 0 + 2 \cdot 3 - 1 \cdot 1 = 3 \neq 0$ So γ is not a solution of $Ax = 0$.

6) Let $v_1 = (1,1,1)$, $v_2 = (1,0,1)$ and $v_3 = (1,1,0)$. It is given that $T = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

(a) Find an orthonormal basis S for $\mathbb{R}^3$ by applying Gram-Schmidt orthogonalization process to the basis T.

$u_1 = (1,1,1)$

$$u_2 = v_2 - \frac{(v_2,u_1)}{(u_1,u_1)} u_1 = (1,0,1) - \frac{1+0+1}{1+1+1}(1,1,1) = (1,0,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

$$u_3 = v_3 - \frac{(v_3,u_1)}{(u_1,u_1)} u_1 - \frac{(v_3,u_2)}{(u_2,u_2)} u_2$$

$$= (1,1,0) - \frac{1+1+0}{1+1+1}(1,1,1) - \frac{\frac{1}{3} - \frac{2}{3} + 0}{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}\left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

$$= (1,1,0) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) + \frac{1}{3} \cdot \frac{9}{6}\left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right) = \left(\frac{1}{2}, 0, \frac{-1}{2}\right)$$

$$w_1 = \frac{u_1}{\|u_1\|} = \frac{(1,1,1)}{\sqrt{1+1+1}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{\left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right)}{\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}} = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$w_3 = \frac{u_3}{\|u_3\|} = \frac{\left(\frac{1}{2}, 0, \frac{-1}{2}\right)}{\sqrt{\frac{1}{4} + 0 + \frac{1}{4}}} = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$$

$S = \{w_1, w_2, w_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Find a basis for the orthogonal complement $W^\perp$ of the subspace $W = Span\{v_1, v_2\}$.

$$W^\perp = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z)\cdot(1,1,1) = 0 \;;\; (x,y,z)\cdot(1,0,1) = 0\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0 \;\&\; x+z=0\}$$

$$\begin{matrix} x+y+z=0 \\ x+z=0 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{-R_2+R_1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{matrix} x+z=0 \\ y=0 \end{matrix} \longrightarrow x = -z.$$

$(x,y,z) = (-z, 0, z) = -z(1,0,-1)$

So $\{(1,0,-1)\}$ is a basis for $W^\perp$.