ÇANKAYA UNIVERSITY

Department of Mathematics

MCS 255 - Vector Calculus and Linear Algebra

SECOND MIDTERM EXAMINATION

12.05.2017

SAMPLE SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: E.M.T.

DURATION: 100 minutes

Question	Grade	Out of
1		18
2		19
3		15
4		15
5		18
6		15
Total		100

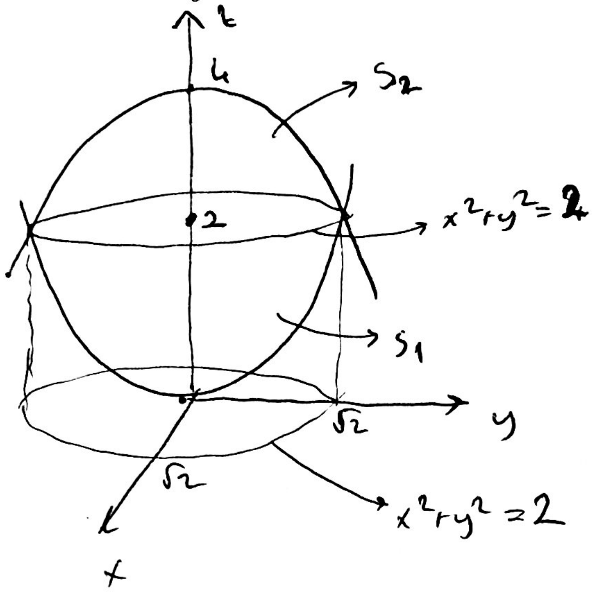
IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Let S be the boundary of the solid region bounded by $z = \quad x^2 + y^2$ and $z = 4 - x^2 - y^2$. Find the area of the surface S.



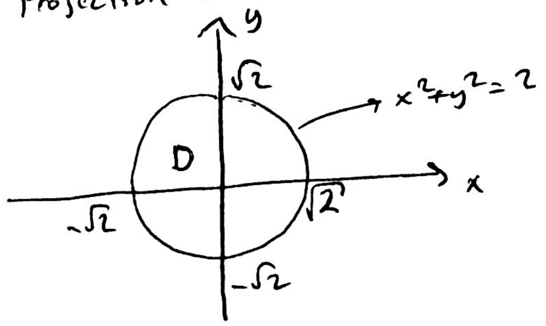
Intersection of surfaces

$z = x^2+y^2$ and $z = 4-x^2-y^2$

$x^2+y^2 = 4-x^2-y^2$

$x^2+y^2 = 2 \longrightarrow z = 2$

Projection on xy-plane is



Surface S_1: $\quad z = x^2+y^2 = f(x,y)$ $\quad f_x = 2x$ $\quad 0 \leq \theta \leq 2\pi$

$(x,y) \in D$. $\quad f_y = 2y$ $\quad 0 \leq r \leq \sqrt{2}$

$$dS = \sqrt{(f_x)^2+(f_y)^2+1}\; dA = \sqrt{4x^2+4y^2+1}\; dA$$

$$\text{Area of } S_1 = \iint_D dS = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$u = \sqrt{4r^2+1}$

$u^2 = 4r^2+1$

$2u\,du = 8r\,dr$

$\frac{1}{4}u\,du = r\,dr$

$r = 0 \to u = \sqrt{4\cdot 0^2+1} = 1$

$r = \sqrt{2} \to u = \sqrt{4\cdot\sqrt{2}^2+1} = 3$

$$= \int_0^{2\pi}\int_1^{3} u\cdot\frac{1}{4}u\,du\,d\theta$$

$$= \int_0^{2\pi} \frac{u^3}{12}\Big|_1^3\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{27}{12}-\frac{1}{12}\right)d\theta = \int_0^{2\pi}\frac{26}{12}\,d\theta = \frac{13}{6}\,\theta\Big|_0^{2\pi} = \frac{13\pi}{3}$$

Surface S_2: $\quad z = 4-x^2-y^2 = g(x,y)$, $(x,y) \in D$ $\quad g_x = -2x$, $g_y = -2y$

$$dS = \sqrt{g_x^2+g_y^2+1}\; dA = \sqrt{4x^2+4y^2+1}\; dA$$

$$\text{Area of } S_2 = \iint_D dS = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\; r\,dr\,d\theta = \frac{13}{3}\pi$$

So Area of S = Area of S_1 + Area of S_2 $= \frac{13\pi}{3} + \frac{13}{3}\pi = \frac{26\pi}{3}$.

2. Use the Divergence Theorem to find the flux $\oiint_S \vec{F} \bullet \vec{n} dS$ of the field

$$\vec{F} = (\cos z^{2017} + x^5 + x^3y^2)\vec{i} + (xe^{-255z} + y^5 + x^2y^3)\vec{j} + (\ln(y^2+1) + 4x^2y^2z + x^{12}y^5)\vec{k}$$

across the surface S where S is the surface of the solid region bounded by the cone $z = \sqrt{x^2+y^2}$ and the paraboloid $z = 2 - x^2 - y^2$.

$\vec{F} = \langle M, N, P \rangle$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N + \frac{\partial}{\partial z} P$$

$$= 5x^4 + 3x^2y^2 + 5y^4 + 3x^2y^2 + 4x^2y^2$$

$$= 5(x^4 + 2x^2y^2 + y^4)$$

$$= 5(x^2+y^2)^2$$

By divergence theorem

$$\oiint_S \vec{F} \bullet \vec{n} dS = \iiint_R (\vec{\nabla} \bullet \vec{F}) dV$$

$$= \iiint_R 5(x^2+y^2)^2 dV$$

$$= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} 5(r^2)^2 r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 5r^5 z \Big|_r^{2-r^2} dr \, d\theta$$

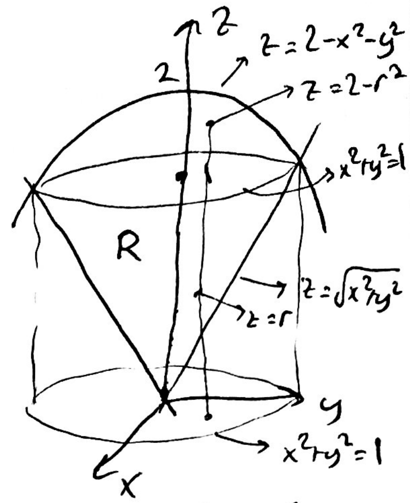
$$= \int_0^{2\pi} \int_0^1 (10r^5 - 5r^7 - 5r^6) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{10}{6} r^6 - \frac{5}{8} r^8 - \frac{5}{7} r^7 \right) \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{5}{3} - \frac{5}{8} - \frac{5}{7} \right) d\theta$$

(56) (21) (24)

$$= \frac{55}{168} \theta \Big|_0^{2\pi} = \frac{55}{84} \pi.$$



Intersection of surfaces

$z = 2 - x^2 - y^2$ & $z = \sqrt{x^2+y^2}$

$z = 2 - z^2$

$z^2 + z - 2 = 0$

$z \quad 2$

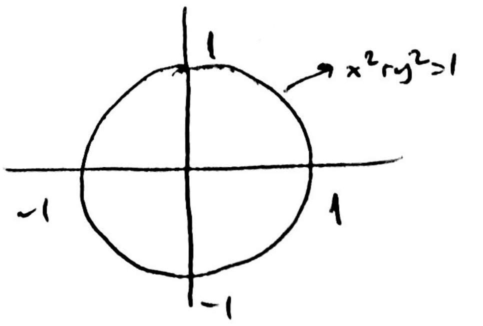
$z \quad -1$

$z = -2$ or $z = 1$ as $z > 0$

$z = 1$

$x^2 + y^2 = 1$

Projection on xy-plane



$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 1$

$r \leq z \leq 2 - r^2$

3. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} x & - & 2y & + & z & - & 2t & = & 1 \\ -2x & + & 4y & - & z & + & t & = & -3 \\ 3x & - & 6y & + & z & & & = & 5 \\ 4x & - & 8y & - & z & + & 7t & = & 9 \end{array}$$

(a) Write the augmented matrix of the system.

(b) Find the reduced row echelon form of the augmented matrix you found in part (a).

(c) Find the set of solutions of the system.

a) $\left[\begin{array}{cccc|c} 1 & -2 & 1 & -2 & 1 \\ -2 & 4 & -1 & 1 & -3 \\ 3 & -6 & 1 & 0 & 5 \\ 4 & -8 & -1 & 7 & 9 \end{array}\right]$

$\downarrow \begin{matrix} 2R_1+R_2 \\ -3R_1+R_3 \\ -4R_1+R_4 \end{matrix}$

b) $\left[\begin{array}{cccc|c} 1 & -2 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & -2 & 6 & 2 \\ 0 & 0 & -5 & 15 & 5 \end{array}\right] \xrightarrow[\substack{2R_2+R_3 \\ 5R_2+R_4}]{-R_2+R_1} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

c) $\begin{matrix} x-2y+t=2 \\ z-3t=-1 \end{matrix} \longrightarrow \begin{matrix} x=2+2y-t \\ z=-1+3t \end{matrix}$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2+2y-t \\ y \\ -1+3t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} t \quad , y, t \in \mathbb{R}.$$

4. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 3 \\ -1 & 4 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$

(a) Find cofactor of each entry of A.

(b) Find $\det(A)$ by using cofactor expansion with respect to second column. Determine whether A is invertible.

(c) Find $Adj(A)$.

(d) Find A^{-1} if A is invertible, by using $Adj(A)$.

(e) Find the solution of $Ax = b$ by using A^{-1} if A is invertible.

a) $c_{11} = (-1)^{1+1}|M_{11}| = \begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix} = 10 - 12 = -2$

$c_{12} = (-1)^{1+2}|M_{12}| = -\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = -(4+3) = -7$

$c_{13} = (-1)^{1+3}|M_{13}| = \begin{vmatrix} 2 & 5 \\ -1 & 4 \end{vmatrix} = 8 + 5 = 13$

$c_{21} = (-1)^{2+1}|M_{21}| = -\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = -(4+4) = -8$

$c_{22} = (-1)^{2+2}|M_{22}| = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 2 - 1 = 1$

$c_{23} = (-1)^{2+3}|M_{23}| = -\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = -(4+2) = -6$

$c_{31} = (-1)^{3+1}|M_{31}| = \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = 6 + 5 = 11$

$c_{32} = (-1)^{3+2}|M_{32}| = -\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = -(3+2) = -5$

$a_{33} = (-1)^{3+3}|M_{33}| = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1$

b) $\det A = a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32}$

$= 2 \cdot (-7) + 5 \cdot 1 + 4 \cdot (-5) = -14 + 5 - 20 = -29$

c) $\operatorname{cof}(A) = \begin{bmatrix} -2 & -7 & 13 \\ -8 & 1 & -6 \\ 11 & -5 & 1 \end{bmatrix}$ $\quad Adj(A) = \operatorname{cof}(A)^T = \begin{bmatrix} -2 & -8 & 11 \\ -7 & 1 & -5 \\ 13 & -6 & 1 \end{bmatrix}$

d) $A^{-1} = \frac{1}{\det A} Adj(A) = \begin{bmatrix} +2/29 & +8/29 & -11/29 \\ +7/29 & -1/29 & +5/29 \\ -13/29 & +6/29 & -1/29 \end{bmatrix}$

e) $X = A^{-1}b = \begin{bmatrix} +2/29 & +8/29 & -11/29 \\ +7/29 & -1/29 & +5/29 \\ -13/29 & +6/29 & -1/29 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} +51/29 \\ +19/29 \\ +2/29 \end{bmatrix}$

5. Let A be a 4×4, B be a 2×4 and C be a 4×2 matrices with $|A| = 3$, $|BC| = 5$ and $|CB| = -1$. In each part, find the given determinant if possible.

a) $|A^5|$.
b) $|2ABC|$.
c) $|A + CB|$.
d) $|(B^T C^T)^{-1} A|$.
e) $|(CBA)^{-1}|$.
f) $|A(B^T C^T)^{-1}|$.

a) $|A^5| = |A|^5 = 3^5 = 243$

b) $|2ABC|$ A is 4×4 BC is 2×2 so ABC is not possible.

c) $|A+CB|$ There is no formula for ~~determinant~~ determinant of the sum.

d) $|(B^T C^T)^{-1} A| = |((CB)^T)^{-1} A| = |(CB)^T|^{-1} |A| = \frac{1}{|(CB)^T|} |A| = \frac{|A|}{|CB|} = \frac{3}{-1}$

$= -3$

e) $|(CBA)^{-1}| = \frac{1}{|CBA|} = \frac{1}{|CB||A|} = \frac{1}{-1 \cdot 3} = -\frac{1}{3}$

f) $|A(B^T C^T)^{-1}| = |A||(B^T C^T)^{-1}|$

$= \frac{|A|}{|B^T C^T|} = \frac{|A|}{|(CB)^T|} = \frac{|A|}{|CB|} = \frac{3}{-1} = -3$

6. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 3 \\ -1 & 4 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$

(a) Find A^{-1} if A is invertible, by using elementary row operations.

(b) Find the solution of $Ax = b$ by using A^{-1} if A is invertible.

(c) Find the solution of $Ax = b$ by using Cramer's Rule if it is possible.

a)
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ -1 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 6 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-6R_2+R_1]{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -11 & 5 & -2 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & -29 & 13 & -6 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{29}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -11 & 5 & -2 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -13/29 & 6/29 & -1/29 \end{array}\right]$$

$$\xrightarrow[-5R_3+R_2]{11R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/29 & 8/29 & -11/29 \\ 0 & 1 & 0 & 7/29 & -1/29 & 5/29 \\ 0 & 0 & 1 & -13/29 & 6/29 & -1/29 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 2/29 & 8/29 & -11/29 \\ 7/29 & -1/29 & 5/29 \\ -13/29 & 6/29 & -1/29 \end{bmatrix}$

b) $x = A^{-1}b = \begin{bmatrix} 2/29 & 8/29 & -11/29 \\ 7/29 & -1/29 & 5/29 \\ -13/29 & 6/29 & -1/29 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 51/29 \\ 19/29 \\ 2/29 \end{bmatrix}$

c) $|A| \overset{-2R_1+R_2}{\underset{R_1+R_3}{=}} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 5 \\ 0 & 6 & 1 \end{vmatrix} \overset{-6R_2+R_3}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & -29 \end{vmatrix} = 1 \cdot 1 \cdot (-29) = -29 \neq 0$

So Cramer's rule is applicable

$|A_1| = \begin{vmatrix} 3 & 2 & -1 \\ 7 & 5 & 3 \\ 1 & 4 & 2 \end{vmatrix} \overset{+3R_1+R_2}{\underset{+2R_1+R_3}{=}} \begin{vmatrix} 3 & 2 & -1 \\ 16 & 11 & 0 \\ 7 & 8 & 0 \end{vmatrix} = (-1)^{1+3}(-1) \begin{vmatrix} 16 & 11 \\ 7 & 8 \end{vmatrix} = -(16 \cdot 8 - 11 \cdot 7)$
$= -128 + 77 = -51$

$|A_2| = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 7 & 3 \\ -1 & 1 & 2 \end{vmatrix} \overset{-2R_1+R_2}{\underset{R_1+R_3}{=}} \begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 5 \\ 0 & 4 & 1 \end{vmatrix} \overset{-4R_2+R_3}{=} \begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & -19 \end{vmatrix} = 1 \cdot 1 \cdot (-19) = -19$

$$|A_3| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -1 & 4 & 1 \end{vmatrix} \underset{R_1+R_3}{\overset{-2R_1+R_2}{=}} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 6 & 4 \end{vmatrix} \overset{-6R_2+R_3}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix} = 1.1.(-2) = -2$$

By Cramer's Rule

$$x = \frac{|A_1|}{|A|} = \frac{-51}{-29} = \frac{51}{29}$$

$$y = \frac{|A_2|}{|A|} = \frac{-19}{-29} = \frac{19}{29}$$

$$z = \frac{|A_3|}{|A|} = \frac{-2}{-29} = \frac{2}{29}$$

$$\text{So } X = \begin{bmatrix} 51/29 \\ 19/29 \\ 2/29 \end{bmatrix}.$$