ÇANKAYA UNIVERSITY

Department of Mathematics

SAMPLE SOLUTIONS

MATH 255 - Vector Calculus and Linear Algebra

FIRST MIDTERM EXAMINATION

30.10.2017

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 100 minutes

Question	Grade	Out of
1		25
2		25
3		25
4		25
Total		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

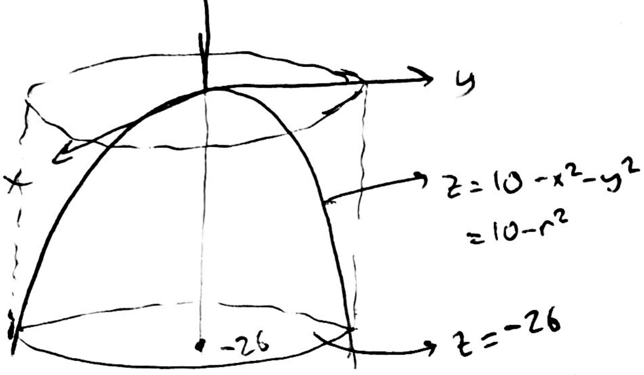
2) Check that the exam paper contains 4 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

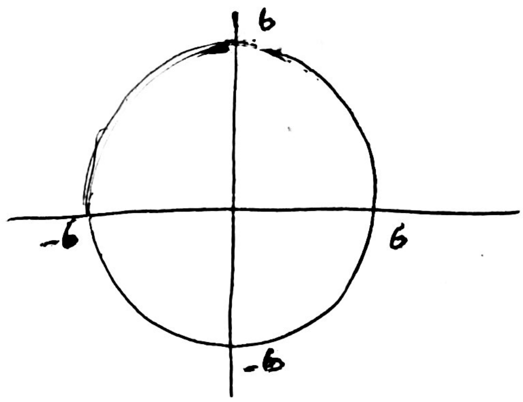
Question 1. a) Let R be the region bounded by $z = 10 - x^2 - y^2$ and $z = -26$. Write the integral $\iiint_R f(x,y,z)\,dV$ as an iterated integral in cylindrical coordinates.

b) Let S be the solid region bounded from below by $x^2 + y^2 + z^2 = 16$ and from above by $z = \sqrt{\frac{1}{3}(x^2+y^2)}$. Express the integral $\iiint_S f(x,y,z)\,dV$ as an iterated integral in spherical coordinates.

a)



$$\left.\begin{aligned} z &= 10 - x^2 - y^2 \\ z &= -26 \end{aligned}\right\} \to x^2 + y^2 = 36$$



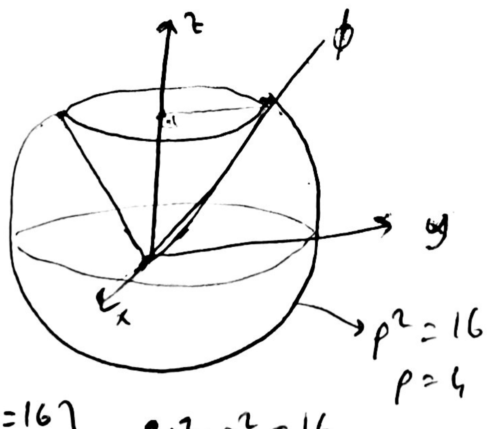
$$\iiint_R f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{6}\int_{-26}^{10-r^2} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 6$

$-26 \leq z \leq 10 - r^2$

b)



$\rho^2 = 16$

$\rho = 4$

$0 \leq \theta \leq 2\pi$

$\frac{\pi}{3} \leq \phi \leq \pi$

$0 \leq \rho \leq 4$

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= 16 \\ 3z^2 &= x^2 + y^2 \end{aligned}\right\}$$

$3z^2 + z^2 = 16$

$z^2 = 4$

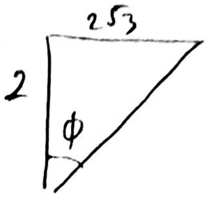
$z = \mp 2 \ , z > 0$

$z = 2$

$x^2 + y^2 = 3 \cdot 2^2 = 12$

$$\iiint_S f(x,y,z)\,dV$$

$$= \int_0^{2\pi}\int_{\pi/3}^{\pi}\int_0^{4} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$



$\tan\phi = \sqrt{3}$

$\phi = \frac{\pi}{3}$

Question 2. Let the force field

$$\vec{F}(x,y,z) = (e^x \cos y + yz^2 + \cos x)\vec{i} + (2 + xz^2 - e^x \sin y)\vec{j} + (2xyz + \frac{1}{z})\vec{k}.$$

be given. Let the curve C be parametrized by

$$\vec{r}(t) = < t^2, t^3\pi, t+1 >, \quad 0 < t < 1.$$

a) By checking the necessary partial derivatives conditions for the force field, show that $\vec{F}$ is conservative.
b) Find a potential function $f(x,y,z)$ for $\vec{F}(x,y,z)$.
c) Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

a) $P(x,y,z) = e^x \cos y + yz^2 + \cos x$

$Q(x,y,z) = 2 + xz^2 - e^x \sin y$

$R(x,y,z) = 2xyz + \frac{1}{z}$

$P_y = -e^x \sin y + z^2 = 0 + z^2 - e^x \sin y = Q_x$

$P_z = 0 + 2yz + 0 = 2yz + 0 = R_x$

$Q_z = 0 + 2xz - 0 = 2xz + 0 = R_y$

So $\vec{F}$ is conservative

b) Let ϕ be a potential for $\vec{F}$. Then

$\phi_x = e^x \cos y + yz^2 + \cos x$

$\phi_y = 2 + xz^2 - e^x \sin y$

$\phi_z = 2xyz + \frac{1}{z}$

$$\phi(x,y,z) = \int (e^x \cos y + yz^2 + \cos x)\, dx$$

$$= e^x \cos y + xyz^2 + \sin x + h(y,z)$$

$2 + xz^2 - e^x \sin y = \phi_y = -e^x \sin y + xz^2 + 0 + h_y(yz)$

$h_y(y,z) = 2$

$h(y,z) = \int 2\, dy = 2y + k(z)$

$\phi(x,y,z) = e^x \cos y + xyz^2 + \sin x + 2y + k(z)$

$2xyz + \frac{1}{z} = \phi_z = 0 + 2xyz + 0 + 0 + k'(z) \rightarrow k'(z) = \frac{1}{z} \rightarrow k(z) = \ln|z| + c$

So $\phi(x,y,z) = e^x \cos y + xyz^2 + \sin x + 2y + \ln|z| + c$ is a potential for $\vec{F}$.

c) By FTLI, $\int_C \vec{F} \cdot d\vec{r} = \phi(1,\pi,2) - \phi(0,0,1) = e^1 \cos\pi + 1 \cdot \pi \cdot 2^2 + \sin 1 + 2\pi + \ln 2 + c$
$- [e^0 \cos 0 + 0 \cdot 0 \cdot 1^2 + \sin 0 + 2 \cdot 0 + \ln 1 + c]$
$= -e + 4\pi + \sin 1 + 2\pi + \ln 2 - 1 = 6\pi + \sin 1 + \ln 2 - e - 1$

Question 3. a) Evaluate the line integral $\int_C \frac{2x^2+y^2}{\sqrt{81y^2+16x^2}}ds$ where C is the part of the ellipse $4x^2+9y^2=36$ in the first quadrant. (Hint: Consider $2x=6\cos t, 3y=6\sin t$).

b) Evaluate the line integral $\int_C y^2dx - z^2dy + x^2dz$, where C is the line segment from $(-1,2,-3)$ to $(3,5,9)$.

a) $x(t) = 3\cos t$ $\quad 0 \leq t \leq \frac{\pi}{2}$

$y(t) = 2\sin t$

$$ds = \sqrt{(x'(t))^2+(y'(t))^2}\,dt = \sqrt{(-3\sin t)^2+(2\cos t)^2}\,dt = \sqrt{9\sin^2 t + 4\cos^2 t}\,dt$$

$$\int_C \frac{2x^2+y^2}{\sqrt{81y^2+16x^2}}ds = \int_0^{\frac{\pi}{2}} \frac{2\cdot 9\cos^2 t + 4\sin^2 t}{\underset{6}{\cancel{\sqrt{81\cdot 4\sin^2 t + 16\cdot 9\cos^2 t}}}} \cdot \cancel{\sqrt{9\sin^2 t + 4\cos^2 t}}\,dt$$

$$= \int_0^{\frac{\pi}{2}} \left(3\cos^2 t + \frac{2}{3}\sin^2 t\right)dt$$

$$= \int_0^{\frac{\pi}{2}} \left(3\,\frac{1+\cos 2t}{2} + \frac{2}{3}\,\frac{1-\cos 2t}{2}\right)dt$$

$$= \left(\frac{3}{2}t + \frac{3}{4}\sin 2t + \frac{1}{3}t - \frac{1}{6}\sin 2t\right)\Big|_0^{\pi/2}$$

$$= \frac{3}{2}\cdot\frac{\pi}{2} + \frac{3}{4}\sin\left(\frac{\pi}{2}\cdot 2\right) + \frac{1}{3}\cdot\frac{\pi}{2} - \frac{1}{6}\sin\left(2\cdot\frac{\pi}{2}\right) - 0$$

$$= \frac{3\pi}{4} + 0 + \frac{\pi}{6} - 0$$

$$= \frac{11\pi}{12}$$

b) C: $\vec{r}(t) = (-1,2,-3) + t\left((3,5,9)-(-1,2,-3)\right) = (-1+4t, 2+3t, -3+12t)$

$x(t) = -1+4t$, $y(t) = 2+3t$, $z(t) = -3+12t$, $0 \leq t \leq 1$

$dx = 4dt$, $dy = 3dt$, $dz = 12dt$

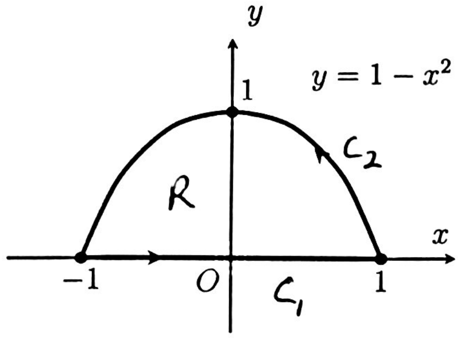
$$\int_C y^2dx - z^2dy + x^2dz = \int_0^1 (2+3t)^2 4dt - (-3+12t)^2 3dt + (-1+4t)^2 12\,dt$$

$$= \int_0^1 \left[16 + 48t + 36t^2 - (27 - 216t + 432t^2) + 12 - 96t + 192t^2\right]dt$$

$$= \int_0^1 (1 + 168t - 204t^2)\,dt = (t + 84t^2 - 68t^3)\Big|_0^1$$

$$= 1 + 84 - 68 - (0+0-0) = 17.$$

Question 4. If $\vec{F} =< 2x\cos x^2 - 5x^2y^2, \ln^3(y+2) + 6x^3y >$, using Green's theorem, evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where C is given in the figure.



$-1 \leq x \leq 1$

$0 \leq y \leq 1-x^2$

C_1 and C_2 are smooth.

So $C = C_1 \cup C_2$ is piecewise smooth, simply closed, positively oriented boundary of R.

$P(x,y) = 2x\cos x^2 - 5x^2y^2$ and $Q(x,y) = \ln^3(y+2) + 6x^3y$

$P_x = 2\cos x^2 + 2x . 2x . (-\sin x^2) - 10xy^2$

$P_y = -10x^2y$

$Q_x = 18x^2y$ and $Q_y = 3\dfrac{\ln^2(y+2)}{y+2} + 6x^3$

are continuous on $y > -1$.

So by Green's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (Q_x - P_y)\,dx\,dy$$

$$= \int_{-1}^{1}\int_0^{1-x^2} \left(18x^2y - (-10x^2y)\right) dy\,dx$$

$$= \int_{-1}^{1}\int_0^{1-x^2} 28x^2y\,dy\,dx$$

$$= \int_{-1}^{1} 14x^2y^2 \Big|_0^{1-x^2} dx = \int_{-1}^{1} 14x^2(1-x^2)^2\,dx$$

$$= \int_{-1}^{1} (14x^2 \; 28x^4 + 14x^6)\,dx$$

$$= \frac{14}{3}x^3 - \frac{28}{5}x^5 + \frac{14}{7}x^7 \Big|_{-1}^{1} = \frac{14}{3} - \frac{28}{5} + \frac{14}{7} - \left(-\frac{14}{3} + \frac{28}{5} - \frac{14}{7}\right)$$

$$= \frac{28}{3} - \frac{56}{5} + 4 = \frac{32}{15}$$