ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 255 - Vector Calculus and Linear Algebra

Final Exam

23.05.2018

Sample Solutions

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: E.M.T.

DURATION: 100 minutes

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

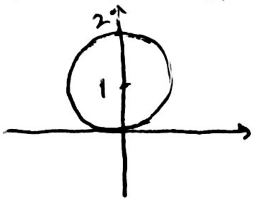
2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Let S be the part of the paraboloid $z = x^2 + y^2$ lying above the region $x^2 + y^2 = 2y$ with downward unit normal vector $\vec{n}$ and C be the boundary curve of S oriented in the clockwise direction when viewed from above (note that orientation of C is positive with respect to $\vec{n}$). Use Stoke's Theorem to evaluate $\oint_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x, y, z) = \langle 3y + z,\ x - z,\ x + 2y \rangle = (3y + z)\vec{\imath} + (x - z)\vec{\jmath} + (x + 2y)\vec{k}$.

$x^2+y^2-2y=0$

$x^2+(y-1)^2=1^2$



D: $0 \le \theta \le \pi$

$0 \le r \le 2\sin\theta$

$r^2-2r\sin\theta=0$

$r=0 \quad r=2\sin\theta$

S is smooth, orientable, C is smooth, closed, positively oriented boundary of S relative to the unit normal vector $\vec{n}$ of S.

$M=3y+z,\ N=x-z,\ P=x+2y$ are polynomials. Hence they have continuous first partial derivatives.

So by Stoke's theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$

A parametrization of S is

$A(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$, $(r,\theta)\in D$

$A_r = \langle \cos\theta, \sin\theta, 2r\rangle$

$A_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

A normal vector to S is $N = A_r \times A_\theta = \begin{vmatrix} \vec{i} & j & k \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle$

So $\vec{n}\,dS = -\dfrac{N}{\|N\|}\|N\|\,dr\,d\theta$

$= \langle 2r^2\cos\theta, 2r^2\sin\theta, -r\rangle\,dr\,d\theta$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y+z & x-z & x+2y \end{vmatrix} = \langle 2-(-1), -(1-1), 1-3\rangle$

$= \langle 3, 0, -2\rangle$

So $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,dS = \int_0^\pi\int_0^{2\sin\theta} \langle 3,0,-2\rangle\cdot\langle 2r^2\cos\theta, 2r^2\sin\theta, -r\rangle\,dr\,d\theta$

$= \int_0^\pi\int_0^{2\sin\theta} (6r^2\cos\theta + 2r)\,dr\,d\theta = \int_0^\pi \left(2r^3\cos\theta\Big|_0^{2\sin\theta} + r^2\Big|_0^{2\sin\theta}\right)d\theta$

$= \int_0^\pi (16\sin^3\theta\cos\theta + 4\sin^2\theta)\,d\theta$

$u=\sin\theta$

$du=\cos\theta\,d\theta$

$\int 16\sin^3\theta\cos\theta\,d\theta = \int 16u^3\,du = 4u^4+c = 4\sin^4\theta+c$

$$= 4\sin^4\theta \Big|_0^{\pi} + \int_0^{\pi} 2(1-\cos 2\theta)\, d\theta$$

$$= 4\sin^4\pi - 4\sin^4 0 + (2\theta - \sin 2\theta)\Big|_0^{\pi}$$

$$= 2\pi - \sin 4\pi - 0 + \sin 0$$

$$= 2\pi.$$

2. This question has FOUR unrelated parts.

Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 & 0 & 7 \\ 1 & 1 & 1 & 1 & 1 & 6 \\ 1 & 1 & 1 & 0 & 1 & 2 \end{bmatrix}$.

(a) Find a basis for the null space of A (i.e. the solution space of the system $Ax = 0$).

(b) Find a basis for the column space of A.

(c) Find a basis for the row space of A.

(d) Find the rank and nullity of A.

$$A \xrightarrow[\substack{-R_1+R_3 \\ -R_1+R_4}]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow[-R_4+R_3]{-R_2+R_1} \begin{array}{c} \begin{array}{cccccc} x & y & z & t & u & w \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix} \end{array}$$

a)
$x + 2z + 3w = 0$ → $x = -2z - 3w$
$y - z + 2w = 0$ → $y = z - 2w$
$t + 4w = 0$ → $t = -4w$
$u - 3w = 0$ → $u = 3w$

$$\begin{pmatrix} x \\ y \\ z \\ t \\ u \\ w \end{pmatrix} = \begin{pmatrix} -2z-3w \\ z-2w \\ z \\ -4w \\ 3w \\ w \end{pmatrix} = z\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} -3 \\ -2 \\ 0 \\ -4 \\ 3 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ -4 \\ 3 \\ 1 \end{bmatrix} \right\}$$

b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

c) $\{ [1\ 0\ 2\ 0\ 0\ 3], [0\ 1\ -1\ 0\ 0\ 2], [0\ 0\ 0\ 1\ 0\ 4], [0\ 0\ 0\ 0\ 1\ -3] \}$

d) $\text{rank}(A) = 4$
$\text{nullity}(A) = 2$.

3. Let $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$.

(a) Show that W is a subspace of $\mathbb{R}^{2\times 2}$.

$0 \in \mathbb{R}$ so $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$ Hence $W \neq \emptyset$.

Let $A, B \in W$, then there are $a, b, c, d, e, f \in \mathbb{R}$ s.t. $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$.

$A + B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix} \in W$ as $a+d, b+e, c+f \in \mathbb{R}$.

Let $k \in \mathbb{R}$.

$kA = \begin{bmatrix} ka & kb \\ 0 & kc \end{bmatrix} \in W$ as $ka, kb, kc \in \mathbb{R}$.

Thus, W is a subspace of $\mathbb{R}^{2\times 2}$.

(b) Find a basis S for W.

$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow a = b = c = 0$

$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for W.

(c) Find a basis B for $\mathbb{R}^{2\times 2}$ containing S.

The standard basis for $\mathbb{R}^{2\times 2}$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & & & \uparrow & \end{matrix} \end{array}$$

So $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

4. This question has TWO unrelated parts.

Let $\langle .\,, . \rangle : \mathbb{P}_2 \times \mathbb{P}_2 \to \mathbb{R}$ be defined by $\langle p(x)\,, q(x)\rangle = \int_1^2 p(x)q(x)dx$ for $p(x)$, $q(x) \in \mathbb{P}_2$.

(a) Show that $\langle .\,, .\rangle$ is an inner product on $\mathbb{P}_2$.

(b) Find an orhthonormal basis for the inner product space $\mathbb{P}_2$ with the inner product $\langle .\,, .\rangle$ defined above by applying the Gram-Schmidt Orhtogonalization Process to the standard basis $\beta = \{1, x, x^2\}$ of $\mathbb{P}_2$.

a) Let $p(x), q(x), r(x) \in \mathbb{P}_2$, $a \in \mathbb{R}$.

$\langle p(x), p(x)\rangle = \int_1^2 p(x)p(x)dx = \int_1^2 (p(x))^2 dx \geq 0$ as $(p(x))^2 \geq 0$.

for all $x \in [1,2]$.

$\langle p(x), p(x)\rangle = 0 \iff \int_1^2 (p(x))^2 dx = 0$

As $1 \neq 2$ and $(p(x))^2 \geq 0$ we have $(p(x))^2 = 0$

which is the case $p(x) = 0$.

$\langle q(x), p(x)\rangle = \int_1^2 q(x)p(x)dx = \int_1^2 p(x)q(x)dx = \langle p(x), q(x)\rangle$.

$$\begin{aligned}\langle p(x)+q(x), r(x)\rangle &= \int_1^2 (p(x)+q(x))r(x)\,dx \\ &= \int_1^2 (p(x)r(x)+q(x)r(x))\,dx \\ &= \int_1^2 p(x)r(x)dx + \int_1^2 q(x)r(x)dx \\ &= \langle p(x), r(x)\rangle + \langle q(x), r(x)\rangle.\end{aligned}$$

$\langle a p(x), q(x)\rangle = \int_1^2 ap(x)q(x)\,dx = a\int_1^2 p(x)q(x)dx = a\langle p(x), q(x)\rangle$.

So $\langle\,,\,\rangle$ is an inner product on $\mathbb{P}_2$.

b) $P_1(x) = 1$

$P_2(x) = x - \frac{\langle P_1(x), x\rangle}{\langle P_1(x), P_1(x)\rangle} P_1(x)$

$= x - \frac{\frac{3}{2}}{1}\cdot 1 = x - \frac{3}{2}$.

$\langle P_1(x), x\rangle = \langle 1, x\rangle = \int_1^2 x\,dx$

$= \frac{x^2}{2}\Big|_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

$\langle P_1(x), P_1(x)\rangle = \langle 1, 1\rangle = \int_1^2 dx = 2-1$

$= 1$

$P_3(x) = x^2 - \frac{\langle P_1(x), x^2\rangle}{\langle P_1(x), P_1(x)\rangle} P_1(x) - \frac{\langle P_2(x), x^2\rangle}{\langle P_2(x), P_2(x)\rangle} P_2(x)$

$\langle P_1(x), x^2\rangle = \langle 1, x^2\rangle$

$= \int_1^2 x^2 dx$

$= \frac{x^3}{3}\Big|_1^2$

$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

$$p_3(x) = x^2 - \frac{\frac{7}{3}}{1} 1 - \frac{\frac{1}{4}}{\frac{1}{12}}\left(x - \frac{3}{2}\right)$$

$$= x^2 - \frac{7}{3} - 3x + \frac{9}{2}$$

$$= x^2 - 3x + \frac{27 - 14}{6}$$

$$= x^2 - 3x + \frac{13}{6}$$

$$\langle p_3(x), p_3(x) \rangle = \int_1^2 \left(x^2 - 3x + \frac{13}{6}\right)^2 dx$$

$$= \int_1^2 \left(x^4 + 9x^2 + \frac{169}{36} - 6x^3 + \frac{13}{3}x^2 - 13x\right) dx$$

$$= \frac{x^5}{5} + 3x^3 + \frac{169}{36}x - \frac{3}{2}x^4 + \frac{13}{9}x^3 - \frac{13}{2}x^2 \Big|_1^2$$

$$= \frac{32}{5} + \cancel{24} + \frac{169}{18} - \cancel{24} + \frac{104}{9} - 26$$

$$- \frac{1}{5} - 3 - \frac{169}{36} + \frac{3}{2} - \frac{13}{9} + \frac{13}{2}$$

$$= \frac{31}{5} + \frac{169}{36} + \frac{91}{9} - 29 + 8$$

$$= \frac{1116 + 845 + 1820}{180} - 21$$

$$= \frac{3781 - 3780}{180} = \frac{1}{180}$$

$$\langle p_2(x), x^2 \rangle = \left\langle x - \frac{3}{2}, x^2 \right\rangle$$

$$= \int_1^2 \left(x - \frac{3}{2}\right) x^2 \, dx$$

$$= \int_1^2 \left(x^3 - \frac{3}{2}x^2\right) dx$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{2}\right) \Big|_1^2$$

$$= \frac{16}{4} - \frac{8}{2} - \frac{1}{4} + \frac{1}{2}$$

$$= \frac{15}{4} - \frac{7}{2} = \frac{1}{4}$$

$$\langle p_2(x), p_2(x) \rangle = \left\langle x - \frac{3}{2}, x - \frac{3}{2} \right\rangle$$

$$= \int_1^2 \left(x^2 - 3x + \frac{9}{4}\right) dx$$

$$= \frac{x^3}{3} - \frac{3}{2}x^2 + \frac{9}{4}x \Big|_1^2$$

$$= \frac{8}{3} - \frac{3}{2} \cdot 4 + \frac{9}{4} \cdot 2 - \frac{1}{3} + \frac{3}{2} - \frac{9}{4}$$

$$= \frac{7}{3} - \frac{9}{2} + \frac{9}{4}$$

$$= \frac{7}{3} - \frac{9}{4} = \frac{28 - 27}{12} = \frac{1}{12}$$

$$q_1(x) = \frac{p_1(x)}{\sqrt{\langle p_1(x), p_1(x) \rangle}} = \frac{1}{1} = 1$$

$$q_2(x) = \frac{p_2(x)}{\sqrt{\langle p_2(x), p_2(x) \rangle}} = \frac{x - \frac{3}{2}}{\frac{1}{12}} = 12x - 18$$

$$q_3(x) = \frac{p_3(x)}{\sqrt{\langle p_3(x), p_3(x) \rangle}} = \frac{x^2 - 3x + \frac{13}{6}}{\frac{1}{180}} = 180x^2 - 540x + 390$$

$\{q_1, q_2, q_3\}$ is an orthonormal basis for $\mathbb{P}_2$.

5. Use Cramer's Rule to solve the system

$$\begin{array}{rcrcrcl} x & + & 2y & + & 5z & = & 2 \\ 2x & - & y & + & 3z & = & 7 \\ -x & - & y & + & 2z & = & 3 \end{array}$$

$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -1 & 3 \\ -1 & -1 & 2 \end{bmatrix}$ $b = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$ $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$Ax = b.$

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 3 \\ -1 & -1 & 2 \end{vmatrix} \xrightarrow[R_1+R_3]{-2R_1+R_2} \begin{vmatrix} 1 & 2 & 5 \\ 0 & -5 & -7 \\ 0 & 1 & 7 \end{vmatrix} = 1.(-1)^{1+1} \begin{vmatrix} -5 & -7 \\ 1 & 7 \end{vmatrix}$$

$$= -35 + 7 = -28 \neq 0$$

So Cramer's rule is applicable.

$$|A_1| = \begin{vmatrix} 2 & 2 & 5 \\ 7 & -1 & 3 \\ 3 & -1 & 2 \end{vmatrix} \overset{2R_2+R_1}{\underset{-R_2+R_3}{=}} \begin{vmatrix} 16 & 0 & 11 \\ 7 & -1 & 3 \\ -4 & 0 & -1 \end{vmatrix} = (-1)(-1)^{2+2} \cdot \begin{vmatrix} 16 & 11 \\ -4 & -1 \end{vmatrix}$$

$$= -(-16+44) = -28$$

$$|A_2| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 7 & 3 \\ -1 & 3 & 2 \end{vmatrix} \overset{-2R_1+R_2}{\underset{R_1+R_3}{=}} \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & -7 \\ 0 & 5 & 7 \end{vmatrix} = 1.(-1)^{1+1} \begin{vmatrix} 3 & -7 \\ 5 & 7 \end{vmatrix}$$

$$= 21 + 35 = 56$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & -1 & 7 \\ -1 & -1 & 3 \end{vmatrix} \overset{-2R_1+R_2}{\underset{R_1+R_3}{=}} \begin{vmatrix} 1 & 2 & 2 \\ 0 & -5 & 3 \\ 0 & 1 & 5 \end{vmatrix} = 1.(-1)^{1+1} \begin{vmatrix} -5 & 3 \\ 1 & 5 \end{vmatrix}$$

$$= -25 - 3 = -28$$

So $x = \dfrac{|A_1|}{|A|} = \dfrac{-28}{-28} = 1$

$y = \dfrac{|A_2|}{|A|} = \dfrac{56}{-28} = -2$

$z = \dfrac{|A_3|}{|A|} = \dfrac{-28}{-28} = 1$ Hence $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$