ÇANKAYA UNIVERSITY

Department of Mathematics

SAMPLE SOLUTIONS

MCS 255 - Vector Calculus and Linear Algebra

SECOND MIDTERM EXAMINATION

19.04.2018

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: E.M.T.

DURATION: 90 minutes

Question	Grade	Out of
1		25
2		25
3		25
4		25
Total		100

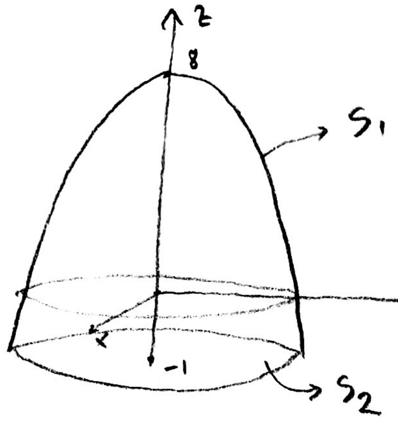
IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 4 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Let S be the boundary surface of the region bounded by the paraboloid $z = 8 - x^2 - y^2$ and the plane $z = -1$. Find the area of S.

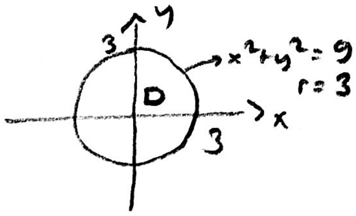


$z = -1$ & $z = 8 - x^2 - y^2$

$-1 = 8 - x^2 - y^2 \rightarrow x^2 + y^2 = 9$

$z = -1$

$S_1 : z = 8 - x^2 - y^2 \qquad (x, y) \in D$



$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 3$

$S_2 : z = -1, \; x = x, \; y = y, \qquad (x, y) \in D.$

S_1: $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$ where $f(x,y) = 8 - x^2 - y^2$

$f_x = -2x, \quad f_y = -2y$

$dS = \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy = \sqrt{4r^2 + 1}\; r\, dr\, d\theta$

$$\text{Area}(S_1) = \iint_{S_1} dS = \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1}\; r\, dr\, d\theta$$

$u = 4r^2 + 1$

$du = 8r\, dr$

$r = 0 \rightarrow u = 1$

$r = 3 \rightarrow u = 37$

$$= \int_0^{2\pi} \int_1^{37} \sqrt{u} \cdot \frac{1}{8}\, du\, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} u^{3/2} \frac{1}{8} \Big|_1^{37} d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} (37\sqrt{37} - 1)\, d\theta$$

$$= \frac{1}{12} (37\sqrt{37} - 1)\, \theta \Big|_0^{2\pi}$$

$$= \frac{\pi}{6} (37\sqrt{37} - 1).$$

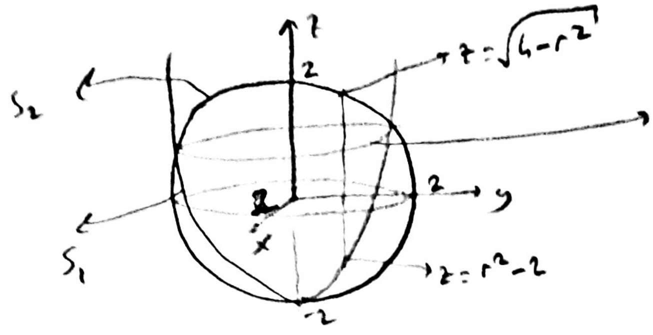
$\text{Area}(S_2) = \pi \cdot 3^2 = 9\pi$

$$\text{Area}(S) = \frac{\pi}{6}(37\sqrt{37} - 1) + 9\pi = \frac{\pi}{6}(37\sqrt{37} + 53).$$

2. Let D be the region inside the sphere $x^2 + y^2 + z^2 = 4$ and above the paraboloid $z = x^2 + y^2 - 2$, S be the boundary of the region D and

$$\vec{F} = \langle x^3 z + xy^2 z,\ -2y^3 z,\ 3y^2 z^2 - x^2 z^2 \rangle .$$

Use the Divergence Theorem to evaluate $\oiint_S \vec{F} \bullet \vec{n} dS$, where $\vec{n}$ is the outward unit notrmal vector to S.



$x^2 + y^2 + z^2 = 4$ & $z = x^2 + y^2 - 2$

$z + 2 + z^2 = 4$

$z^2 + z - 2 = 0$

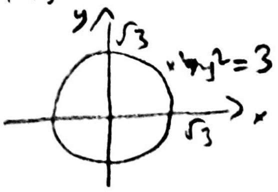
$(z+2)(z-1) = 0$

$z = -2$ or $z = 1$

$\downarrow$

$x^2 + y^2 = 3$

Projection on xy-plane



$x^2 + y^2 = 3 \to r = \sqrt{3}$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq \sqrt{3}$

$r^2 - 2 \leq z \leq \sqrt{4 - r^2}$

S_1 and S_2 are smooth, orientable. So $S = S_1 \cup S_2$ is piecewise smooth, orientable, closed surface.

Components of $\vec{F}$ are polynomials, so they have continuous first partial derivatives.

$$\vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^3 z + xy^2 z,\ -2y^3 z,\ 3y^2 z^2 - x^2 z^2 \rangle$$

$$= 3x^2 z + y^2 z - 6y^2 z + 6y^2 z - 2x^2 z$$

$$= x^2 z + y^2 z = (x^2 + y^2) z = r^2 z$$

By divergence thm

$$\oiint_S \vec{F} \cdot \vec{n} dS = \iiint_D \vec{\nabla} \cdot \vec{F} dV = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2-2}^{\sqrt{4-r^2}} r^2 z \, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} r^3 \frac{z^2}{2} \Big|_{r^2-2}^{\sqrt{4-r^2}} dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{r^3}{2} \left(4 - r^2 - (r^2 - 2)^2 \right) dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{r^3}{2} \left(\cancel{4} - r^2 - r^4 + 4r^2 - \cancel{4} \right) dr \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \int_0^{\sqrt{3}} (3r^5 - r^7)\, dr\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\frac{r^6}{2} - \frac{r^8}{8} \right) \Big|_0^{\sqrt{3}} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\frac{(\sqrt{3})^6}{2} - \frac{(\sqrt{3})^8}{8} \right) d\theta$$

$$= \frac{1}{2} \left[\frac{27}{2} - \frac{81}{8} \right] \theta \Big|_0^{2\pi}$$

$$= \frac{1}{2} \left[\frac{108 - 81}{8} \right] 2\pi$$

$$= \frac{27}{8} \pi.$$

3. Let $A = \begin{bmatrix} 3 & -4 & -1 \\ 2 & 1 & 3 \\ -2 & 12 & k \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

(a) Find the values of k for which the matrix A is invertible.

(b) Find A^{-1} by using elementary row operations if $k = 11$ and solve the system $Ax = b$.

a) $$\left[\begin{array}{ccc|ccc} 3 & -4 & -1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ -2 & 12 & k & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & -1 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 13 & k+3 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & -1 & 0 \\ 0 & 11 & 11 & -2 & 3 & 0 \\ 0 & 13 & k+3 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{11}R_2} \left[\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & -1 & 0 \\ 0 & 1 & 1 & -\frac{2}{11} & \frac{3}{11} & 0 \\ 0 & 13 & k+3 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[-13R_2+R_3]{5R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{11} & \frac{4}{11} & 0 \\ 0 & 1 & 1 & -\frac{2}{11} & \frac{3}{11} & 0 \\ 0 & 0 & k-10 & \frac{26}{11} & -\frac{28}{11} & 1 \end{array}\right]$$

A is invertible if and only if $k-10 \neq 0$.

So, $k \in \mathbb{R} \setminus \{10\}$.

b) If $k=11$, then

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{11} & \frac{4}{11} & 0 \\ 0 & 1 & 1 & -\frac{2}{11} & \frac{3}{11} & 0 \\ 0 & 0 & 1 & \frac{26}{11} & -\frac{28}{11} & 1 \end{array}\right] \xrightarrow[-R_3+R_2]{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{25}{11} & \frac{32}{11} & -1 \\ 0 & 1 & 0 & -\frac{28}{11} & \frac{31}{11} & -1 \\ 0 & 0 & 1 & \frac{26}{11} & -\frac{28}{11} & 1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -\frac{25}{11} & \frac{32}{11} & -1 \\ -\frac{28}{11} & \frac{31}{11} & -1 \\ \frac{26}{11} & -\frac{28}{11} & 1 \end{bmatrix}$

$Ax = b \longrightarrow A^{-1}(Ax) = A^{-1}b \longrightarrow x = A^{-1}b$

So $x = A^{-1}b = \begin{bmatrix} -\frac{25}{11} & \frac{32}{11} & -1 \\ -\frac{28}{11} & \frac{31}{11} & -1 \\ \frac{26}{11} & -\frac{28}{11} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is the unique solution of the system

4. Given the system of linear equations

$$\begin{array}{rcrcrcrcrcr} x & - & 2y & + & z & + & 4u & - & 2w & = & -1 \\ 2x & - & 4y & + & 3z & + & 9u & - & 4w & = & -6 \\ -3x & + & 6y & - & 3z & - & 12u & + & 7w & = & 2 \\ 4x & - & 8y & + & 4z & + & 16u & - & 8w & = & -4 \end{array}$$

(a) Write the augmented matrix of the system.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 4 & -2 & -1 \\ 2 & -4 & 3 & 9 & -4 & -6 \\ -3 & 6 & -3 & -12 & 7 & 2 \\ 4 & -8 & 4 & 16 & -8 & -4 \end{array}\right]$$

(b) Convert the augmented matrix to its reduced row echelon form.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 4 & -2 & -1 \\ 2 & -4 & 3 & 9 & -4 & -6 \\ -3 & 6 & -3 & -12 & 7 & 2 \\ 4 & -8 & 4 & 16 & -8 & -4 \end{array}\right] \xrightarrow[-4R_1+R_4]{\substack{-2R_1+R_2 \\ 3R_1+R_3}} \left[\begin{array}{ccccc|c} 1 & -2 & 1 & 4 & -2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & -2 & 3 \\ 0 & 0 & 1 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{2R_3+R_1} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is the reduced row echelon form of the augmented matrix.

(c) Solve the system.

$x - 2y + 3u = 1$

$z + u = -4$

$w = -1$

$\longrightarrow$

$x = 1 + 2y - 3u$

$z = -4 - u$

$w = -1$

$$\begin{pmatrix} x \\ y \\ z \\ u \\ w \end{pmatrix} = \begin{pmatrix} 1 + 2y - 3u \\ y \\ -4 - u \\ u \\ -1 \end{pmatrix} \qquad y, u \in \mathbb{R}.$$