ÇANKAYA UNIVERSITY

Department of Mathematics

MCS 255 - Vector Calculus and Linear Algebra

SAMPLE SOLUTIONS

FIRST MIDTERM EXAMINATION

15.03.2018

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: E.M.T.

DURATION: 90 minutes

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT NOTES:

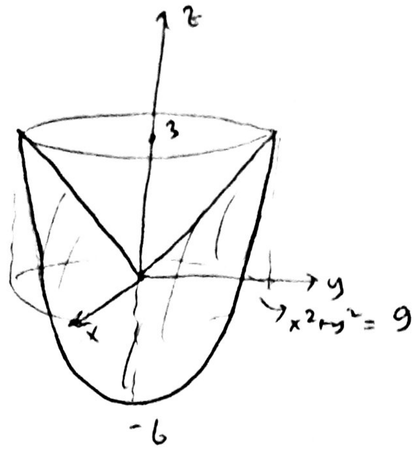
1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Let R be the region bounded by the paraboloid $z = x^2 + y^2 - 6$ and the cone $z = \sqrt{x^2 + y^2}$. Use cylindrical coordinates to evaluate the triple integral

$$\iiint_R dxdydz$$



$z^2 = x^2 + y^2$

$x^2 + y^2 = z + 6$

$z^2 = z + 6$

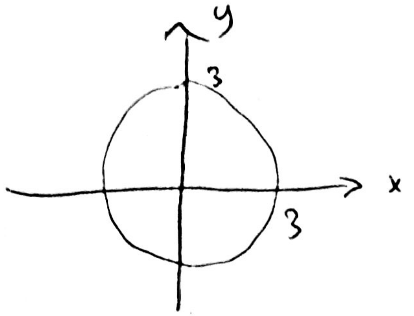
$z^2 - z - 6 = 0$

$z \quad -3$

$z \quad 2$

$z = 3$ or $z = -2$ & $z > 0$

$z = 3.$



$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 3$

$r^2 - 6 \leq z \leq r$

$$\iiint_R dxdydz = \int_0^{2\pi}\int_0^3\int_{r^2-6}^{r} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 rz \Big|_{r^2-6}^{r} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (r^2 - r^3 + 6r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^3}{3} - \frac{r^4}{4} + 3r^2\right)\Big|_0^3 d\theta$$

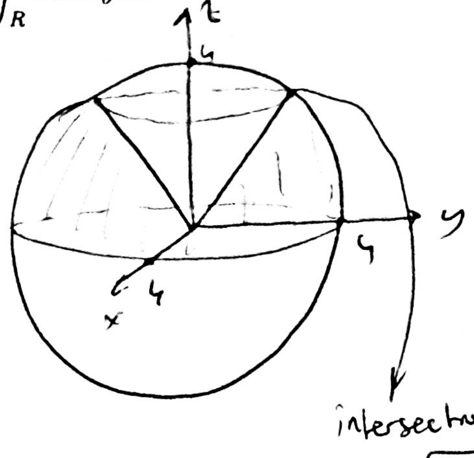
$$= \int_0^{2\pi}\left(\frac{3^3}{3} - \frac{81}{4} + 3\cdot 3^2\right) d\theta$$

$$= \int_0^{2\pi} \frac{63}{4}\,d\theta$$

$$= \frac{63}{4}\,\theta\Big|_0^{2\pi}$$

$$= \frac{63}{2}\pi$$

2. Let R be the region bounded by the xy-plane, the cone $z = \sqrt{3x^2 + 3y^2}$ and the sphere $x^2 + y^2 + z^2 = 16$. Use spherical coordinates to evaluate the triple integral $\iiint_R z\, dxdydz$.



$$0 \le \theta \le 2\pi$$

$$\frac{\pi}{6} \le \phi \le \frac{\pi}{2}$$

$$0 \le \rho \le 4$$

intersection of $z = \sqrt{3x^2 + 3y^2}$

and $x^2 + y^2 + z^2 = 16$

$$3x^2 + 3y^2 = z^2 = 16 - x^2 - y^2$$

$$x^2 + y^2 = 4$$

$$z^2 = 12 \rightarrow z = \mp 2\sqrt{3} \ \& \ z > 0$$

$$z = 2\sqrt{3}$$

$$\rho^2 \sin^2\phi = 4 \rightarrow \rho \sin\phi = 2$$

$$\rho \cos\phi = 2\sqrt{3}$$

$$\tan\phi = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{\pi}{6}$$

$$\iiint_R z\, dxdydz = \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^4 \rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \frac{\rho^4}{4} \sin\phi\cos\phi \Big|_0^4 d\phi\, d\theta$$

$$= \int_0^{2\pi} 64 \frac{\sin^2\phi}{2} \Big|_{\pi/6}^{\pi/2} d\theta$$

$$= \frac{64}{2}\left(\sin^2\frac{\pi}{2} - \sin^2\frac{\pi}{6}\right) \theta \Big|_0^{2\pi}$$

$$= 32\left(1 - \frac{1}{4}\right) \cdot 2\pi = 48\pi$$

3. Let C be the line segment from the point $P(1,3,5)$ to the point $Q(-11,7,2)$. Evaluate the line integral $\int_C (xy+yz)\,ds$

$\vec{r}(t) = (1,3,5) + t\,((-11,7,2) - (1,3,5))$

$= (1-12t,\ 3+4t,\ 5-3t)\ , \quad 0\le t\le 1$

$\frac{d\vec{r}}{dt} = \langle -12, 4, -3\rangle$

$ds = \left\|\frac{d\vec{r}}{dt}\right\| dt = \sqrt{(-12)^2+4^2+(-3)^2}\,dt = 13\,dt$

$\int_C (xy+yz)\,ds = \int_0^1 [(1-12t)(3+4t) + (3+4t)(5-3t)]\,13\,dt$

$= \int_0^1 (3-36t+4t-48t^2+15-9t+20t-12t^2)\,13\,dt$

$= \int_0^1 (-60t^2-21t+18)\,13\,dt$

$= 13\left(-20t^3-\frac{21}{2}t^2+18t\right)\Big|_0^1$

$= 13\cdot\left(-20-\frac{21}{2}+18\right)$

$= -\frac{13\cdot 25}{2} = -\frac{325}{2}.$

4. Let C be the curve with parametrization

$$\vec{r}(t) = \left\langle 1 + (t-1)e^{t^{15}}, 2t + \cos(\frac{\pi t}{2}), \sin(\frac{\pi t}{2}) \right\rangle, \quad 0 \le t \le 1.$$

Evaluate the line integral $\int_C \vec{F} \bullet d\vec{r}$ where

$$\vec{F}(x,y,z) = \langle 2xy^3z^4 + 2\cos x, \quad \sin y + 3x^2y^2z^4, \quad e^z + 4x^2y^3z^3 \rangle$$

$M(x,y,z) = 2xy^3z^4 + 2\cos x$
$N(x,y,z) = \sin y + 3x^2y^2z^4$
$P(x,y,z) = e^z + 4x^2y^3z^3$

$M_y = 6xy^2z^4 = N_x$
$M_z = 8xy^3z^3 = P_x$
$N_z = 12x^2y^2z^3 = P_y$ So $\vec{F}$ is conservative, let $\phi(x,y,z)$ be a potential function for $\vec{F}(x,y,z)$. Then

$\phi_x = M = 2xy^3z^4 + 2\cos x$
So $\phi(x,y,z) = \int(2xy^3z^4 + 2\cos x)dx = x^2y^3z^4 + 2\sin x + g(y,z)$.
$\sin y + 3x^2y^2z^4 = \phi_y = 3x^2y^2z^4 + g_y(y,z)$
$g_y(y,z) = \sin y \longrightarrow g(y,z) = \int \sin y \, dy = -\cos y + h(z)$
$\phi(x,y,z) = x^2y^3z^4 + 2\sin x - \cos y + h(z)$
$e^z + 4x^2y^3z^3 = \phi_z = 4x^2y^3z^3 + h'(z)$
$h'(z) = e^z \longrightarrow h(z) = \int e^z dz = e^z + c$
So $\phi(x,y,z) = x^2y^3z^4 + 2\sin x - \cos y + e^z + c$ is a potential for $\vec{F}$ for $c \in \mathbb{R}$.

By the FTLI,

$\vec{r}(1) = \langle 1 + 0e^1, 2 + \cos\frac{\pi}{2}, \sin\frac{\pi}{2} \rangle$
$= \langle 1, 2, 1 \rangle$
$\vec{r}(0) = \langle \cancel{1} + (-1)e^0, 0 + \cos 0, \sin 0 \rangle$
$= \langle 0, 1, 0 \rangle$

$\int_C \vec{F} \cdot d\vec{r} = \phi(\vec{r}(1)) - \phi(\vec{r}(0))$

$= \phi(1,2,1) - \phi(0,1,0)$

$= 1^2 \cdot 2^3 \cdot 1^4 + 2\sin 1 - \cos 2 + e^1 + c - (0 + 2\sin 0 - \cos 1 + e^0 + c)$

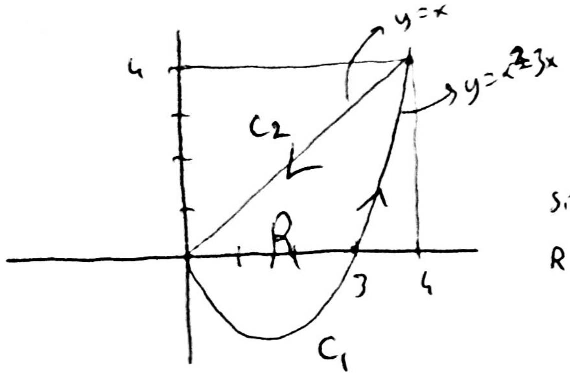
$= 8 + 2\sin 1 - \cos 2 + e + \cos 1 - 1$

$= 7 + 2\sin 1 - \cos 2 + e + \cos 1.$

5. Let C_1 be the part of the parabola $y = x^2 - 3x$ from the point $(0,0)$ to the point $(4,4)$ and C_2 be the line segment from the point $(4,4)$ to the point $(0,0)$. If $C = C_1 \cup C_2$ and

$$\vec{F}(x,y) = \left\langle e^{x^{15}} - 2y + \cos(x^3),\quad \sin(y^{2018}) + x^2 - \ln(y^2+1) \right\rangle$$

evaluate $\oint_C \vec{F} \bullet d\vec{r}$



C_1 and C_2 are both smooth

So $C = C_1 \cup C_2$ is piecewise smooth, simple closed, positively oriented boundary of R,

$M = e^{x^{15}} - 2y + \cos(x^3)$, $N = \sin(y^{2018}) + x^2 - \ln(y^2+1)$

$M_x = 15x^4 e^{x^{15}} - 3x^2 \sin(x^3)$, $M_y = -2$, $N_x = 2x$, $N_y = 2018 y^{2017} \cos(y^{2018}) - \frac{2y}{y^2+1}$

are continuous everywhere.

So by Green's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y)\, dA \qquad R: 0 \le x \le 4,\ x^3 - 3x \le y \le x$$

$$= \iint_R (2x+2)\, dx\, dy$$

$$= \int_0^4 \int_{x^2-3x}^{x} (2x+2)\, dy\, dx$$

$$= \int_0^4 (2x+2)\, y \Big|_{x^2-3x}^{x} dx$$

$$= \int_0^4 (2x+2)(x - x^2 + 3x)\, dx$$

$$= \int_0^4 (-2x^3 + 8x^2 - 2x^2 + 8x)\, dx$$

$$= -\frac{2}{4}x^4 + \frac{8}{3}x^3 - \frac{2}{3}x^3 + 4x^2 \Big|_0^4$$

$$= -\frac{2}{\cancel{4}}\, 4^{\cancel{4}\,3} + 2 \cdot 4^3 + 4 \cdot 4^2$$

$$= -128 + 128 + 64 = 64$$