Çankaya University
Department of Mathematics
2018 - 2019 Fall Semester

MATH 255 - Vector Calculus and Linear Algebra

Final Examination

1) For the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 6 & 0 & 1 \\ 4 & 6 & 6 & 1 \end{bmatrix}$

a) Find a basis for the nullspace.

b) Find the rank and nullity of A.

2) Use Gram-Schmidt orthogonalization process to obtain an orthonormal basis for $\mathbb{R}^3$ starting with:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$$

3) Find an orthogonal matrix P that diagonalizes $A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$.

4) Find all eigenvalues and the corresponding eigenvectors for the matrix $A = \begin{bmatrix} 2 & 0 & -5 \\ 2 & 3 & 10 \\ 0 & 0 & 3 \end{bmatrix}$.

5) Find the values of c such that the set of vectors $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} c \\ 1 \\ 4 \\ 5 \end{bmatrix} \right\}$

a) is linearly independent.

b) spans $\mathbb{R}^4$.

6) Consider the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \\ a-2b \\ c \\ 3c \end{bmatrix}$.

a) Find a basis for this subspace.

b) Find dimension of this subspace.

c) Express the vector $\overrightarrow{u} = \begin{bmatrix} 5 \\ 0 \\ 7 \\ -9 \\ 1 \\ 3 \end{bmatrix}$ in terms of the basis you found in part **a)**.

Answers

1) a) RREF is: $\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

This is equivalent to the system:
$$\begin{aligned} x_1 + 6x_3 &= 0 \\ x_2 - 3x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

There are infinitely many solutions. x_3 is a free parameter:

$$\begin{aligned} x_4 &= 0 \\ x_2 &= 3x_3 \\ x_1 &= -6x_3 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_3 \\ 3x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

Therefore a basis for the nullspace is: $A = \left\{ \begin{bmatrix} -6 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

b) rank(A) is the number of leading 1's.

rank$(A) = 3$, nullity $(A) = 4 - 3 = 1$.

2) $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \overrightarrow{u_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \overrightarrow{u_3} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \overrightarrow{v_2} &= \overrightarrow{u_2} - \frac{\langle \overrightarrow{u_2}, \overrightarrow{v_1} \rangle}{\langle \overrightarrow{v_1}, \overrightarrow{v_1} \rangle} \overrightarrow{v_1} \\ &= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{2+1}{1+1+1} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \overrightarrow{v_3} &= \overrightarrow{u_3} - \frac{\langle \overrightarrow{u_3}, \overrightarrow{v_1} \rangle}{\langle \overrightarrow{v_1}, \overrightarrow{v_1} \rangle} \overrightarrow{v_1} - \frac{\langle \overrightarrow{u_3}, \overrightarrow{v_2} \rangle}{\langle \overrightarrow{v_2}, \overrightarrow{v_2} \rangle} \overrightarrow{v_2} \\ &= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Normalization:

$$\overrightarrow{w_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{w_2} = \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{w_3} = \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

3) $\det(\lambda I - A) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda - 3 & 2 \\ 2 & \lambda - 6 \end{vmatrix} = 0$

Characteristic equation is:

$(\lambda - 3)(\lambda - 6) - 4 = 0$

$\lambda^2 - 9\lambda + 14 = 0$

Eigenvalues are: $\lambda_1 = 2, \quad \lambda_2 = 7$.

- $\lambda_1 = 2$

  $$\begin{aligned} x - 2y &= 0 \\ -2x + 4y &= 0 \end{aligned}$$

  One solution is: $\overrightarrow{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- $\lambda_2 = 7$

  $$\begin{aligned} -4x - 2y &= 0 \\ -2x - y &= 0 \end{aligned}$$

  One solution is: $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

If we normalize these vectors we get $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ therefore:

$P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

4) $\det(\lambda I - A) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda - 2 & 0 & 5 \\ -2 & \lambda - 3 & -10 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = 0$

Characteristic equation is:

$(\lambda - 2)(\lambda - 3)^2 = 0$

Eigenvalues are: $\lambda_1 = 2, \quad \lambda_{2,3} = 3.$

- $\lambda = 2$

  $\Rightarrow \quad x_3 = 0, \quad x_1 + \dfrac{x_2}{2} = 0$

  One solution is: $\overrightarrow{v_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$

- $\lambda = 3$

  $\Rightarrow \quad x_1 + 5x_3 = 0$

  One solution is: $\overrightarrow{v_2} = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}.$

  Another solution is: $\overrightarrow{v_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

Eigenvectors are: $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

5) a) Let $A = \begin{bmatrix} 1 & 0 & 1 & c \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 5 \end{bmatrix}$

$\det(A) = 4c - 16$

If $\det(A) \neq 0$, the matrix is invertible, therefore only the trivial solution exists for the homogeneous equation and the given vectors are linearly independent.

In other words, $c \neq 4$.

b) Once again we start with the same matrix.

For $c \neq 4$, the determinant is nonzero, A is invertible and therefore the given set spans $\mathbb{R}^4$.

6) a) $\begin{bmatrix} a \\ 0 \\ b \\ a-2b \\ c \\ 3c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$

Therefore a basis is: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}.$

b) Dimension is the number of vectors in the basis. Therefore dim$= 3$.

c) $\overrightarrow{u} = \begin{bmatrix} 5 \\ 0 \\ 7 \\ -9 \\ 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$

$\Rightarrow \quad c_1 = 5, \quad c_2 = 7, \quad c_3 = 1$