Çankaya University
Department of Mathematics
2019 - 2020 Fall Semester

MATH 255 - Vector Calculus and Linear Algebra

Final Examination

1) The following set of vectors S span $\mathbb{R}^4$ but S is not a basis for $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Remove some vectors from S such that remaining set is a basis for $\mathbb{R}^4$.

2) Find a basis for nullspace, rank and nullity of the matrix $A = \begin{bmatrix} 1 & 3 & 0 & -1 & 0 \\ 3 & 9 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 & -2 \\ 4 & 12 & 2 & -2 & -4 \end{bmatrix}$.

3) Use Gram-Schmidt orthogonalization process to obtain an orthonormal basis for span(S):

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 10 \\ 5 \\ 4 \end{bmatrix} \right\}$$

4) Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -9 & 0 & -10 \\ 2 & 1 & 2 \\ 5 & 0 & 6 \end{bmatrix}$.

5) Let $B = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$.

a) Find an orthogonal matrix P that diagonalizes B.

b) Find B^9.

Answers

1) Consider the matrix A whose columns are vectors in S:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 2 \\ 2 & -1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 4 & 3 & 0 & 2 & 1 \end{bmatrix}$$

After row operations, we obtain the REF (Row-Echelon Form) as:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The positions of leading 1's give us the basis vectors. In other words, we have to choose the first, second,, fourth and sixth vectors to form a basis:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

2) After row operations, we obtain the RREF (Reduced Row-Echelon Form) of the matrix as:

$$\begin{bmatrix} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 3x_2 - x_4 &= 0 \\ x_3 + x_4 - 2x_5 &= 0 \end{aligned}$$

There are 3 leading 1's, therefore rank of A is 2.

There are 3 free parameters (x_2, x_4, x_5) in the solution of $A\overrightarrow{x} = \overrightarrow{0}$, therefore the nullity of A is 3.

The solution to homogeneous equation is: $x_1 = -3x_2 + x_4, \quad x_3 = -x_4 + 2x_5$. Therefore a basis for nullspace is:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3) The given vectors are:

$$\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \overrightarrow{u_2} = \begin{bmatrix} -1 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \quad \overrightarrow{u_3} = \begin{bmatrix} 2 \\ 10 \\ 5 \\ 4 \end{bmatrix}.$$

Let's use the Gram-Schmidt process:

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\langle \overrightarrow{u_2}, \overrightarrow{v_1} \rangle}{\langle \overrightarrow{v_1}, \overrightarrow{v_1} \rangle} \overrightarrow{v_1}$$

$$= \begin{bmatrix} -1 \\ 4 \\ 2 \\ 3 \end{bmatrix} - \frac{-1+6}{1+4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\overrightarrow{v_3} = \overrightarrow{u_3} - \frac{\langle \overrightarrow{u_3}, \overrightarrow{v_1} \rangle}{\langle \overrightarrow{v_1}, \overrightarrow{v_1} \rangle} \overrightarrow{v_1} - \frac{\langle \overrightarrow{u_3}, \overrightarrow{v_2} \rangle}{\langle \overrightarrow{v_2}, \overrightarrow{v_2} \rangle} \overrightarrow{v_2}$$

$$= \begin{bmatrix} 2 \\ 10 \\ 5 \\ 4 \end{bmatrix} - \frac{2+8}{1+4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} - \frac{50}{25} \begin{bmatrix} -2 \\ 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ -2 \end{bmatrix}$$

After normalization, we obtain:

$$\overrightarrow{w_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\overrightarrow{w_2} = \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|} = \frac{1}{5} \begin{bmatrix} -2 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\overrightarrow{w_3} = \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|} = \frac{1}{5} \begin{bmatrix} 4 \\ 2 \\ 1 \\ -2 \end{bmatrix}$$

4) $\det(\lambda I - A) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda+9 & 0 & 10 \\ -2 & \lambda-1 & -2 \\ -5 & 0 & \lambda-6 \end{vmatrix} = 0$

Characteristic equation is:

$$\begin{aligned} (\lambda-1)\Big[(\lambda+9)(\lambda-6)+50\Big] &= 0 \\ (\lambda-1)\Big[\lambda^2+3\lambda-4\Big] &= 0 \\ (\lambda-1)^2(\lambda+4) &= 0 \end{aligned}$$

Eigenvalues are: $\lambda_1 = 1$ (double root) and $\lambda_2 = -4$.

- $\lambda_1 = 1$

$$\begin{aligned} 10x + 10z &= 0 \\ -2x - 2z &= 0 \\ -5x - 5y &= 0 \end{aligned}$$

Note that all three equations are the same. y and z are free parameters and $x = -z$. We can find two linearly independent eigenvectors:

$$\overrightarrow{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \overrightarrow{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

- $\lambda_2 = -4$

$$\begin{aligned} 5x + 10z &= 0 \\ -2x - 5y - 2z &= 0 \\ -5x - 10z &= 0 \end{aligned}$$

We can easily see that $x = -2z$ and $y = \frac{2}{5}z$. A solution is:

$$\overrightarrow{v_3} = \begin{bmatrix} -10 \\ 2 \\ 5 \end{bmatrix}.$$

5) a) $\left|\lambda I - B\right| = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda - 1 & -4 \\ -4 & \lambda - 1 \end{vmatrix} = 0$

$\lambda^2 - 2\lambda - 15 = 0 \quad \Rightarrow \quad (\lambda - 5)(\lambda + 3) = 0$

Eigenvalues are: $\lambda = 5$ and $\lambda = -3$.

- $\lambda = 5$

$$\begin{array}{rcl} 4x - 4y & = & 0 \\ -4x + 4y & = & 0 \end{array} \quad \Rightarrow \quad \text{An eigenvector is: } \overrightarrow{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $\lambda = -3$

$$\begin{array}{rcl} -4x - 4y & = & 0 \\ -4x - 4y & = & 0 \end{array} \quad \Rightarrow \quad \text{An eigenvector is: } \overrightarrow{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

After normalization, we obtain the orthogonal matrix as: $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

b) $P^{-1} = P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

$$\begin{aligned} D &= P^{-1}BP = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 5 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \end{aligned}$$

$$B = PDP^{-1} \quad \Rightarrow \quad B^9 = PD^9P^{-1}$$

$$\begin{aligned} B^9 &= P^{-1}BP = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5^9 & 0 \\ 0 & -3^9 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 5^9 - 3^9 & 5^9 + 3^9 \\ 5^9 + 3^9 & 5^9 - 3^9 \end{bmatrix} \end{aligned}$$