Çankaya University
Department of Mathematics
2018 - 2019 Fall Semester

MATH 255 - Vector Calculus and Linear Algebra

First Midterm Examination

1) Find the volume of the solid below the paraboloids $z = x^2 + y^2 - 9$ and $z = 1 - \dfrac{x^2 + y^2}{9}$.

2) Evaluate $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $\overrightarrow{F} = (3x + 5y)\overrightarrow{i} + (x - 8y)\overrightarrow{j}$ and C is the arc of the circle $x^2 + (y-1)^2 = 1$ from $(1,1)$ to $(0,2)$.

3) This question has two unrelated parts:

a) Determine if the following vector field is conservative or not: $\overrightarrow{F} = 6xy\,\overrightarrow{i} + \left(3x^2 + 5\right)\overrightarrow{j}$

b) Evaluate $\displaystyle\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $\overrightarrow{F} = (y^3 + x^2)\overrightarrow{i} + (x^3 + y^4)\overrightarrow{j}$ and C is the unit circle.

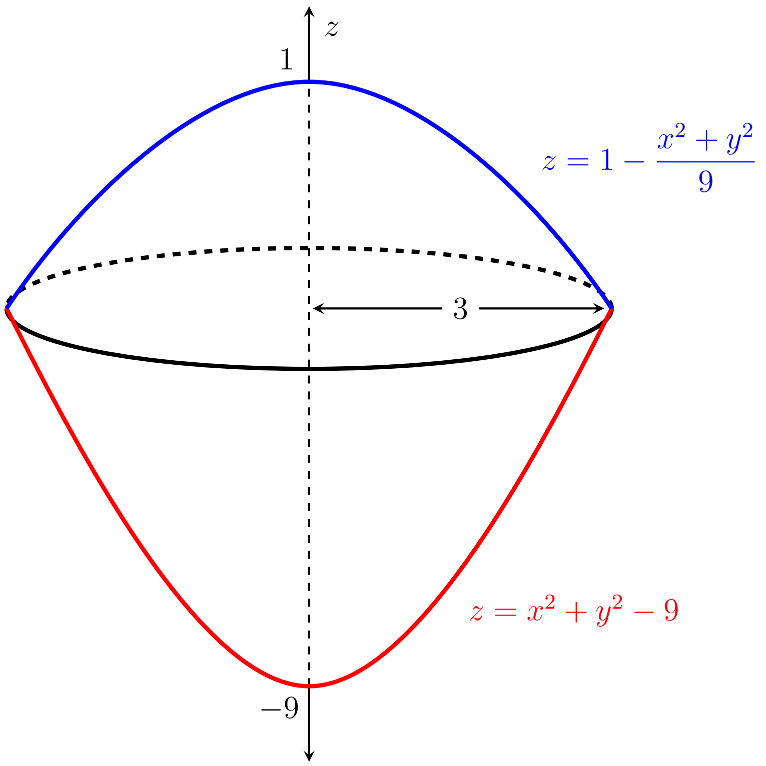
4) Consider the paraboloid $z = x^2 + y^2 + 4$. Find the area of the part of this paraboloid below the plane $z = 8$.

5) Evaluate either $\displaystyle\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS$ or $\displaystyle\iiint_D \overrightarrow{\nabla} \cdot \overrightarrow{F}\, dV$ where $\overrightarrow{F} = (xz + y^2)\overrightarrow{i} + (z^2 - x^2)\overrightarrow{j} + (3xy)\overrightarrow{k}$, D is the solid cylinder bounded by $x^2 + y^2 = 9,\ z = 0,\ z = 4$ and S is the surface of D.

Answers

1) The paraboloids intersect at: $x^2 + y^2 - 9 = 1 - \dfrac{x^2 + y^2}{9}$

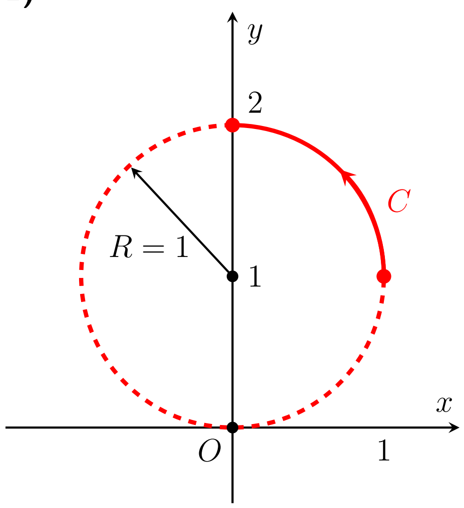
$10(x^2 + y^2) = 90 \quad \Rightarrow \quad x^2 + y^2 = 9, \quad r = 3.$ So the figure is:



Using cylindrical coordinates, we can find the volume as:

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^3\int_{r^2-9}^{1-\frac{r^2}{9}} dz\, r dr\, d\theta \\
&= \int_0^{2\pi}\int_0^3 \left(1 - \frac{r^2}{9} - r^2 + 9\right) r dr\, d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^3 \left(10r - \frac{10r^3}{9}\right) dr \\
&= \theta\Big|_0^{2\pi} \cdot \left(5r^2 - \frac{10r^4}{36}\right)\Big|_0^3 \\
&= 2\pi \cdot \frac{45}{2} \\
&= 45\pi
\end{aligned}$$

2)



$$C: \begin{array}{rcl} x & = & \cos t \\ y & = & 1+\sin t \end{array}, \quad 0 \leqslant t \leqslant 2\pi$$

$$\overrightarrow{F} = \big(3\cos t + 5 + 5\sin t\big)\overrightarrow{i} + \big(\cos t - 8 - 8\sin t\big)\overrightarrow{j}$$

$$d\overrightarrow{r} = \big(-\sin t\,\overrightarrow{i} + \cos t\,\overrightarrow{j}\big)dt$$

$$\overrightarrow{F}\cdot d\overrightarrow{r} = \cos^2 t - 5\sin^2 t - 5\sin t - 8\cos t - 11\sin t\cos t$$

$$\begin{aligned}
\int_C \overrightarrow{F}\cdot d\overrightarrow{r} &= \int_0^{\pi/2} \Big(\cos^2 t - 5\sin^2 t - 5\sin t - 8\cos t - 11\sin t\cos t\Big)dt \\
&= \int_0^{\pi/2} \left(\frac{1+\cos(2t)}{2} - 5\,\frac{1-\cos(2t)}{2} - 5\sin t - 8\cos t - 11\,\frac{\sin(2t)}{2}\right) dt \\
&= -2t + \frac{3}{2}\sin(2t) + 5\cos t - 8\sin t + \frac{11}{4}\cos(2t)\Big|_0^{\pi/2} \\
&= -\pi - \frac{37}{2}
\end{aligned}$$

3) a)

$$\begin{aligned} \frac{\partial}{\partial y}\big(6xy\big) &= 6x \\ \frac{\partial}{\partial x}\big(3x^2+5\big) &= 6x \end{aligned} \quad \Rightarrow \quad \overrightarrow{F} \text{ is conservative.}$$

b) Using Green's Theorem: $\oint Pdx + Qdy = \iint_D (Q_x - P_y)dA$, we obtain:

$$\begin{aligned} \iint_D (3x^2 - 3y^2)dA &= \int_0^{2\pi}\int_0^1 3r^2\big(\cos^2\theta - \sin^2\theta\big)\, rdr\, d\theta \\ &= \int_0^{2\pi} \cos(2\theta)\, d\theta \;\cdot\; \int_0^1 3r^3\, dr \\ &= \frac{\sin(2\theta)}{2}\bigg|_0^{2\pi} \;\cdot\; \frac{3}{4}\, r^4\bigg|_0^1 \\ &= 0 \end{aligned}$$

4) The equation of the surface is: $z = f(x,y) = x^2 + y^2 + 4$

$$\Rightarrow \quad f_x = 2x, \quad f_y = 2y \quad \Rightarrow \quad \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2 + 1}$$

$$z = 8 \quad \Rightarrow \quad x^2 + y^2 + 4 = 8 \quad \Rightarrow \quad x^2 + y^2 = 4 \quad \Rightarrow \quad r = 2$$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 \sqrt{4r^2 + 1}\, r dr$$

Using substitution $u = 4r^2 + 1,\ du = 8rdr$

$$= \int_0^{2\pi} d\theta \ \cdot \ \int_1^{17} \sqrt{u}\, \frac{du}{8}$$

$$= \theta \Big|_0^{2\pi} \ \cdot \ \frac{1}{8} \cdot \frac{2}{3}\, u^{3/2} \Big|_1^{17}$$

$$= \frac{\pi}{6}\left(17^{3/2} - 1\right)$$

5) $\overrightarrow{\nabla} \cdot \overrightarrow{F} = \dfrac{\partial}{\partial x}(xz + y^2) + \dfrac{\partial}{\partial y}(z^2 - x^2) + \dfrac{\partial}{\partial z}(3xy) = z$

$$x^2 + y^2 = 9 \quad \Rightarrow \quad r = 3$$

$$\iiint_D \overrightarrow{\nabla} \cdot \overrightarrow{F}\, dV = \iiint_D z\, dV$$

$$= \int_0^{2\pi} \int_0^3 \int_0^4 z\, dz\, r dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 8\, r dr\, d\theta$$

$$= \int_0^{2\pi} 36\, d\theta$$

$$= 72\pi$$