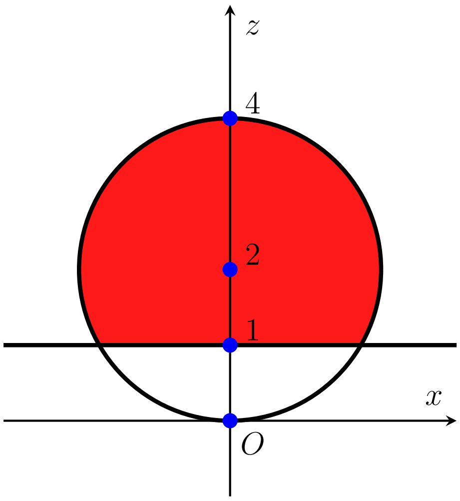
Çankaya University
Department of Mathematics
2019 - 2020 Fall Semester

MATH 255 - Vector Calculus and Linear Algebra

First Midterm Examination

1) Find the volume of the solid bounded by the sphere $x^2+y^2+(z-2)^2=4$ and the plane $z=1$.



2) Find $\int_C f(x,y)\,ds$ where $f(x,y)=2x-y$ and C is the line segment between the points $(3,4)$ and $(8,14)$.

3) Find $\oint_C \overrightarrow{F}\cdot d\overrightarrow{r}$ where $\overrightarrow{F}=\left(6xy^2-y\right)\overrightarrow{i}+\left(6x^2y+x^3\right)\overrightarrow{j}$ and C is the unit circle.

4) Find the surface area of the part of the paraboloid $z=\frac{1}{8}-2x^2-2y^2$ above the $xy-$plane.

5) Evaluate either $\iint_S \overrightarrow{F}\cdot\overrightarrow{n}\,dS$ or $\iiint_D \overrightarrow{\nabla}\cdot\overrightarrow{F}\,dV$ where:

$$\overrightarrow{F}=\left(xyz+3x^2\right)\overrightarrow{i}+\left(xy+yz^2-2zy\right)\overrightarrow{j}+\left(z^2-6xz\right)\overrightarrow{k},$$

D is the rectangular prism $0\leqslant x\leqslant 1,\quad 0\leqslant y\leqslant 2,\quad 0\leqslant z\leqslant 3$ and S is the surface of D.

Answers

1)

Spherical Coordinates:

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{2\pi/3}\int_0^{2} \rho^2 d\rho\, \sin\phi d\phi\, d\theta + \frac{1}{3}\pi(\sqrt{3}\,)^2\cdot 1 \\
&= \frac{8}{3}\int_0^{2\pi}\int_0^{2\pi/3} \sin\phi d\phi\, d\theta + \frac{1}{3}\pi(\sqrt{3}\,)^2\cdot 1 \\
&= \frac{8}{3}\left(-\cos\frac{2\pi}{3} + \cos 0\right)\int_0^{2\pi} d\theta + \pi \\
&= \frac{8}{3}\left(1+\frac{1}{2}\right)2\pi + \frac{1}{3}\pi(\sqrt{3}\,)^2\cdot 1 \\
&= 8\pi + \pi \\
&= 9\pi
\end{aligned}$$

Cylindrical Coordinates:

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{2+\sqrt{4-r^2}} dz\, rdr\, d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_{2-\sqrt{4-r^2}}^{2+\sqrt{4-r^2}} dz\, rdr\, d\theta \\
&= 2\pi\left[\int_0^{\sqrt{3}}\int_1^{2+\sqrt{4-r^2}} dz\, rdr + \int_{\sqrt{3}}^{2}\int_{2-\sqrt{4-r^2}}^{2+\sqrt{4-r^2}} dz\, rdr\right] \\
&= 2\pi\left[\int_0^{\sqrt{3}} \left(1+\sqrt{4-r^2}\,\right) rdr + \int_{\sqrt{3}}^{2} \left(2\sqrt{4-r^2}\,\right) rdr\right] \\
&= 2\pi\left[\frac{r^2}{2} - \frac{1}{3}\left(4-r^2\right)^{3/2}\right]\Bigg|_0^{\sqrt{3}} + 2\pi\left[-\frac{2}{3}\left(4-r^2\right)^{3/2}\right]\Bigg|_{\sqrt{3}}^{2} \\
&= 2\pi\left[\frac{3}{2} - \frac{1}{3} + \frac{8}{3}\right] + 2\pi\left[\frac{2}{3}\right] \\
&= \frac{23\pi}{3} + \frac{4\pi}{3} \\
&= 9\pi
\end{aligned}$$

2) The parametrization for the line segment is:

$$C: \begin{array}{rcl} x & = & 3+5t \\ y & = & 4+10t \end{array}, \quad 0 \leqslant t \leqslant 1$$

$$ds = \sqrt{5^2+10^2}\,dt = \sqrt{125}\,dt = 5\sqrt{5}\,dt$$

$$\begin{aligned} \int_C f(x,y)\,ds &= \int_0^1 \Big(6+10t-4-10t\Big)\,5\sqrt{5}\,dt \\ &= \int_0^1 10\sqrt{5}\,dt \\ &= 10\sqrt{5}\,t\,\Big|_0^1 \\ &= 10\sqrt{5} \end{aligned}$$

3) Let's use Green's Theorem:

$$Q = 6x^2y + x^3 \quad \Rightarrow \quad \frac{\partial Q}{\partial x} = 12xy + 3x^2$$

$$P = 6xy^2 - y \quad \Rightarrow \quad \frac{\partial P}{\partial y} = 12xy - 1$$

$$\begin{aligned}
\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \iint_D \left(3x^2 + 1\right) dA \\
&= \int_0^{2\pi} \int_0^1 \left(3r^2 \cos^2\theta + 1\right) r dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 \left(3r^3 \cos^2\theta + r\right) dr\, d\theta \\
&= \int_0^{2\pi} \left(\frac{3}{4}\cos^2\theta + \frac{1}{2}\right) d\theta \\
&= \frac{7\pi}{4}
\end{aligned}$$

Alternative Solution: Let's evaluate the integral directly:

$$C : \quad \begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned} \qquad 0 \leqslant t \leqslant 2\pi$$

$$\overrightarrow{F} = \left(6\cos t \sin^2 t - \sin t\right)\overrightarrow{i} + \left(6\cos^2 t \sin t + \cos^3 t\right)\overrightarrow{j}$$

$$r = \cos t\,\overrightarrow{i} + \sin t\,\overrightarrow{j} \quad \Rightarrow \quad d\overrightarrow{r} = -\sin t\,\overrightarrow{i} + \cos t\,\overrightarrow{j}$$

$$\begin{aligned}
\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^{2\pi} \Big(-6\cos t \sin^3 t + \sin^2 t + 6\cos^3 t \sin t + \cos^4 t \Big) dt \\
&= \left(-\frac{3\sin^4 t}{2} + \frac{t}{2} - \frac{\sin 2t}{4} - \frac{3\cos^4 t}{2} + \frac{3t}{8} + \frac{\sin 2t}{4} + \frac{\sin 4t}{32} \right) \Bigg|_0^{2\pi} \\
&= \frac{7\pi}{4}
\end{aligned}$$

4) $z = f(x,y) = \frac{1}{8} - 2x^2 - 2y^2$

$f_x = -4x, \quad f_y = -4y \quad \Rightarrow \quad \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{16x^2 + 16y^2 + 1}$

$2r^2 = \frac{1}{8} \quad \Rightarrow \quad r = \frac{1}{4}$

$$
\begin{aligned}
S &= \iint_S dS \\
&= \iint_A \sqrt{16x^2 + 16y^2 + 1}\, dA \\
&= \int_0^{2\pi} \int_0^{1/4} \sqrt{16r^2 + 1}\, r dr\, d\theta \\
&= \frac{1}{48} \int_0^{2\pi} (16r^2 + 1)^{3/2} \Big|_0^{1/4} d\theta \\
&= \frac{1}{48} \big(2\sqrt{2} - 1\big) \int_0^{2\pi} d\theta \\
&= \frac{\big(2\sqrt{2} - 1\big)\pi}{24}
\end{aligned}
$$

5)
$$\begin{aligned}
\overrightarrow{\nabla}\cdot\overrightarrow{F} &= \frac{\partial}{\partial x}\left(xyz+3x^2\right)+\frac{\partial}{\partial y}\left(xy+yz^2-2zy\right)+\frac{\partial}{\partial z}\left(z^2-6xz\right)\\
&= yz+6x+x+z^2-2z+2z-6x\\
&= x+yz+z^2
\end{aligned}$$

$$\begin{aligned}
\iiint_D \overrightarrow{\nabla}\cdot\overrightarrow{F}\,dV &= \iiint_V \left(x+yz+z^2\right)dV\\
&= \int_0^1\int_0^2\int_0^3 \left(x+yz+z^2\right)dz\,dy\,dx\\
&= \int_0^1\int_0^2 \left(xz+\frac{yz^2}{2}+\frac{z^3}{3}\right)\Bigg|_0^3 dy\,dx\\
&= \int_0^1\int_0^2 \left(3x+\frac{9y}{2}+9\right)dy\,dx\\
&= \int_0^1 \left(3xy+\frac{9y^2}{4}+9y\right)\Bigg|_0^2 dx\\
&= \int_0^1 (6x+27)\,dx\\
&= \left(3x^2+27x\right)\Bigg|_0^1\\
&= 30
\end{aligned}$$

Alternative Solution: $\iint_S \overrightarrow{F}\cdot\overrightarrow{n}\,dS$ must be evaluated on 6 surfaces. But on 3 of them, $(x=0, y=0, z=0)$ there is no contribution because $\overrightarrow{F}=0$.

$$\begin{aligned}
\iint_S \overrightarrow{F}\cdot\overrightarrow{n}\,dS &= \int_0^2\int_0^3 (yz+3)\,dzdy+\int_0^1\int_0^3 \left(2x+2z^2-4z\right)dzdx+\int_0^1\int_0^2 (9-18x)\,dydx\\
&= 27+3+0\\
&= 30
\end{aligned}$$