Çankaya University
Department of Mathematics
2018 - 2019 Fall Semester

MATH 255 - Vector Calculus and Linear Algebra

Second Midterm Examination

1) Consider the following linear system of equations:

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + z &= 1 \\ -6x + 7y - z &= -10 \end{aligned}$$

a) Evaluate the inverse of the coefficient matrix A^{-1}.

b) Use the inverse of the coefficient matrix, A^{-1}, to solve the system.

2) Solve the following linear system of equations:

$$\begin{aligned} x_1 + 2x_3 - 3x_4 &= 7 \\ -2x_1 + x_2 - 3x_3 + 7x_4 &= -11 \\ 4x_1 - 2x_2 + 7x_3 - 10x_4 &= 25 \end{aligned}$$

3) This question has two unrelated parts:

a) Evaluate the determinant of $A = \begin{bmatrix} 3 & 1 & 5 & 1 \\ 0 & 8 & 4 & 2 \\ -3 & 7 & -2 & 4 \\ 3 & 9 & 9 & 7 \end{bmatrix}$.

b) Does each of the following sets represent a subspace for the corresponding vector space? Verify your answer by calculation.

1. $\begin{bmatrix} a \\ b \\ b \\ a \end{bmatrix}$
2. $\begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}$
3. The set of functions that satisfy $\displaystyle\int_0^1 f(x)\,dx = 2$
4. The set of polynomials of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where $a_2 = a_1 a_3$

4) Consider the following set of vectors $A = \left\{ \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\}$.

a) Does A span $\mathbb{R}^3$? Verify your answer.

b) Are the vectors in A linearly independent? Verify your answer.

c) Is A a basis for $\mathbb{R}^3$?

5) Find a basis for $\mathbb{R}^3$ choosing vectors from the set $A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix} \right\}$.

Answers

1) a) $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 1 \\ -6 & 7 & -1 \end{bmatrix}$

$$A^{-1} = \frac{\mathsf{adj}\ A}{\mathsf{det}\ A} = \frac{\begin{bmatrix} -7 & -5 & 7 \\ 31 & 22 & -32 \\ 3 & -2 & -3 \end{bmatrix}^T}{|A|} = \begin{bmatrix} 7 & -31 & -3 \\ 5 & -22 & 2 \\ -7 & 32 & 3 \end{bmatrix}$$

b) $A\overrightarrow{x} = \overrightarrow{b} \quad \Rightarrow \quad \overrightarrow{x} = A^{-1}\overrightarrow{b}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & -31 & -3 \\ 5 & -22 & 2 \\ -7 & 32 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -10 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix}$$

2) The augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 7 \\ -2 & 1 & -3 & 7 & -11 \\ 4 & -2 & 7 & -10 & 25 \end{array}\right]$$

After elementary row operations we obtain:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -11 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 3 \end{array}\right]$$

This is equivalent to the system:

$$\begin{aligned} x_1 - 11x_4 &= 1 \\ x_2 - 3x_4 &= 0 \\ x_3 + 4x_4 &= 3 \end{aligned}$$

There are infinitely many solutions. x_4 is a free parameter:

$$\begin{aligned} x_4 &= t \\ x_3 &= 3 - 4t \\ x_2 &= 3t \\ x_1 &= 1 + 11t \end{aligned}$$

3) a) Let's reduce A to upper triangular form.

After $R_3 \to R_3 + R_1$ and $R_4 \to R_4 - R_1$ we obtain:

$$\begin{vmatrix} 3 & 1 & 5 & 1 \\ 0 & 8 & 4 & 2 \\ 0 & 8 & 3 & 5 \\ 0 & 8 & 4 & 6 \end{vmatrix}$$

After $R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_2$ we obtain:

$$\begin{vmatrix} 3 & 1 & 5 & 1 \\ 0 & 8 & 4 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$\Rightarrow \quad \det(A) = 3 \cdot 8 \cdot (-1) \cdot 4 = -96$$

b)

1. YES. $\begin{bmatrix} a_1 \\ b_1 \\ b_1 \\ a_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ b_2 \\ a_2 \end{bmatrix} \in W$ and $k \begin{bmatrix} a_1 \\ b_1 \\ b_1 \\ a_1 \end{bmatrix} \in W$.

2. NO. We can show this with a counterexample:

   $\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \in W$ but $5 \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 0 \end{bmatrix} \notin W$.

3. NO. We can show this with a counterexample:

   $f(x) = 2 \in W$ but $3f(x) = 6 \notin W$.

4. NO. We can show this with a counterexample:

   $1 + x + x^2 + x^3 \in W$ but $2 + 2x + 2x^2 + 2x^3 \notin W$.

4) a) Given $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, can we find $c_1,\ c_2, c_3$ such that

$$c_1 \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

It is sufficient to check determinant. Determinant of the coefficient matrix is $-2 \neq 0$, therefore it is invertible. The system has unique solution for all x, y, z, so A spans $\mathbb{R}^3$.

b) Can we find nonzero $c_1,\ c_2, c_3$ such that

$$c_1 \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Once again, it is sufficient to check determinant. It is the same matrix as part **a)**. The determinant is nonzero, matrix is invertible. Therefore the system has only the trivial solution $c_1 = 0,\ c_2 = 0,\ c_3 = 0$, so A is linearly independent.

c) Using parts **a)** and **b)**, we see that A spans $\mathbb{R}^3$ and A is linearly independent. Therefore A is a basis for $\mathbb{R}^3$.

5) Consider the matrix whose columns are the vectors in A: $\begin{bmatrix} 1 & 5 & -1 & 2 \\ 1 & -3 & 1 & -4 \\ 1 & -2 & -1 & -5 \end{bmatrix}$

After row reduction, we find: $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

The first three columns contain leading 1's and they are linearly independent. Therefore the first three columns of the original matrix are also linearly independent and form a basis for $\mathbb{R}^3$.

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$