Çankaya University
Department of Mathematics
2019 - 2020 Fall Semester

MATH 255 - Vector Calculus and Linear Algebra

Second Midterm Examination

1) For which values of a and b does the following system of equations have

a) Unique solution?

b) Infinitely many solutions?

c) No solution?

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2 \\ x_2 + (3a+2)x_3 &= 3b+3 \\ 2x_1 + 3x_2 + (4a+6)x_3 &= 4b+7 \end{aligned}$$

2) Let $A = \begin{bmatrix} 2 & 3 & -3 \\ 5 & -4 & 8 \\ 10 & 3 & 1 \end{bmatrix}$.

a) Find the determinant of A.

b) Find the inverse of A.

3) Consider the given vector spaces V and their subsets W. Is W a subspace of V? Explain.

a) $V = \mathbb{R}^5$, W is the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \\ 0 \\ c \end{bmatrix}$ where $a + b + c = 12$.

b) $V = M_{2\times 3}$, W is the set of matrices of the form $\begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$.

c) $V = P_3$, W is the set of polynomials of the form $a + bx^2 + (a + 2b)x^3$.

4) The following sets are subsets of the vector space $\mathbb{R}^4$.

a) Is $S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$ linearly independent?

b) Does $S_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$ span $\mathbb{R}^4$?

5) Consider the following homogeneous system of 2 equations in 5 unknowns:

$$\begin{aligned} x_1 + 3x_2 - x_4 &= 0 \\ 2x_1 + 6x_2 + x_3 - 2x_4 - 2x_5 &= 0 \end{aligned}$$

a) Find a basis for solution space.

b) Find the dimension of solution space.

Answers

1) The augmented matrix is: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 3a+2 & 3b+3 \\ 2 & 3 & 4a+6 & 4b+7 \end{array}\right]$

After elementary row operations we obtain: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a & b \end{array}\right]$

The last row of the matrix should be checked for different possibilities:

- $a \neq 0 \quad \Rightarrow$ Unique solution.
- $a = 0,\ b = 0 \quad \Rightarrow$ Infinitely many solutions.
- $a = 0,\ b \neq 0 \quad \Rightarrow$ No solution.

2) a) $|A| = 4$

b) $A^{-1} = \dfrac{1}{4}\begin{bmatrix} -28 & -12 & 12 \\ 75 & 32 & -31 \\ 55 & 24 & -23 \end{bmatrix}$

3) a) NO. For example:

$$\overrightarrow{u} = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \\ 4 \end{bmatrix} \in W, \quad \overrightarrow{v} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 6 \end{bmatrix} \in W \quad \text{but} \quad \overrightarrow{u} + \overrightarrow{v} = \begin{bmatrix} 10 \\ 0 \\ 4 \\ 0 \\ 10 \end{bmatrix} \notin W$$

b) NO. For example:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \in W, \quad \text{but} \quad 2M = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 8 & 18 \end{bmatrix} \notin W$$

c) YES. Let $p = a_1 + b_1x^2 + (a_1 + 2b_1)x^3,\ q = a_2 + b_2x^2 + (a_2 + 2b_2)x^3$. Then:

$$p + q = (a_1 + a_2) + (b_1 + b_2)x^2 + \big(a_1 + a2 + 2(b_1 + b_2)\big)x^3 \in W$$

$$kp = ka_1 + kb_1x^2 + k(a_1 + 2b_1)x^3 \in W$$

4) a) Consider the equation: $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Is there a nontrivial solution? We have to check the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right]$

After row operations, we obtain: $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

We can see that $c_3 = 0, \quad \Rightarrow \quad c_2 = 0, \quad \Rightarrow \quad c_1 = 0$. Therefore S_1 is Linearly Independent.

b) Consider the equation: $c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix} + c_5 \begin{bmatrix} 2 \\ 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Is there always a solution for given numbers x_1, x_2, x_3, x_4?

Consider the augmented matrix: $\left[\begin{array}{ccccc|c} 0 & 2 & 0 & 4 & 2 & x_1 \\ 0 & 1 & 0 & 2 & 1 & x_2 \\ 1 & 1 & 0 & 4 & 4 & x_3 \\ 1 & 0 & 1 & 2 & 0 & x_4 \end{array}\right]$

After row operations, we obtain: $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & x_4 \\ 0 & 1 & -1 & 2 & 4 & x_3 - x_4 \\ 0 & 0 & 1 & 0 & -3 & x_2 - x_3 + x_4 \\ 0 & 0 & 0 & 0 & 0 & x_1 - 2x_2 \end{array}\right]$

Checking last row, we see that there is a solution if and only if $x_1 = 2x_2$. Therefore S_2 does NOT span $\mathbb{R}^4$.

5) The reduced row-echelon form of the augmented matrix is:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{array}\right]$$

The equations give the results: $x_3 = 2x_5, \quad x_1 = -3x_2 + x_4$. In vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Therefore a basis is:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Dimension is 3.