ÇANKAYA UNIVERSITY

Department of Mathematics

- SOLUTIONS -

MCS 255 - Vector Calculus and Linear Algebra

SECOND MIDTERM EXAMINATION

04.12.2017

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: **B.K. & E.M.T.**

DURATION: 100 minutes

Question	Grade	Out of
1		25
2		25
3		25
4		25
Total		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 4 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Consider the system of linear equations

$$\begin{array}{rcrcrcl} x & & & + & 2z & = & 4 \\ 2x & + & y & + & (a+4)z & = & a^2+8 \\ x & + & 2y & + & (a^2+2a-7)z & = & 2a^2+a+7 \end{array}$$

Find the values of "a" for which the system has

(a) no solution,

(b) infinitely many solutions,

(c) a unique solution.

Augmented Matrix of the system $\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 2 & 1 & (a+4) & a^2+8 \\ 1 & 2 & (a^2+2a-7) & 2a^2+a+7 \end{array}\right]$

$$\xrightarrow[-R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & a & a^2 \\ 0 & 2 & a^2+2a-9 & 2a^2+a+3 \end{array}\right] \xrightarrow{-2R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & a & a^2 \\ 0 & 0 & a^2-9 & a+3 \end{array}\right]$$

a) Since all the rows, except the 3rd row, contain "Leading 1's", the only inconsistency may occur in the 3rd row resulting in "no solution". So if; $a^2-9=0 \Rightarrow a=3$ or $a=-3$, $a+3 \neq 0 \Rightarrow a \neq -3$ } So if $\boxed{a=3}$, 3rd row gives inconsistency $(0 \neq 0)$,

$\Rightarrow$ hence the system has $\boxed{\text{no solution if } a=3}$.

b) If the 3rd row is a "zero row", that is if $a^2-9=0$ and $a+3=0$
$\Rightarrow$ if $\boxed{a=-3}$, then there are infinitely many solutions.

c) If $a^2-9 \neq 0$, i.e. $\boxed{\text{if } a \neq 3 \text{ and } a \neq -3}$, then the coefficient matrix can be row reduced to identity, hence the system will have a unique solution.

2. Given the system of linear equations

$$\begin{array}{rcrcrcr} x & + & 4y & + & 2z & = & 1 \\ & & 2y & - & z & = & -1 \\ x & + & 2y & + & z & = & 4 \end{array}$$

corresponding to the matrix equation $A\vec{x} = \vec{b}$ where A is the coefficient matrix, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

(a) Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 2 & -1 & -1 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

(b) Find A^{-1} by using **elementary row operations**.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_2+R_3 \to R_3]{-2R_2+R_1 \to R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3+R_2 \to R_2]{2R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 2 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3 \to R_3]{\frac{1}{2}R_2 \to R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \underbrace{}_{I} \; \underbrace{}_{A^{-1}}$$

$$\Rightarrow \text{So, } \boxed{A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}}$$

(c) Use A^{-1} found in part (b) to solve the system $A\vec{x} = \vec{b}$.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -1 & 0 & 2 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 7 \\ -1 \\ -1 \end{bmatrix}}_{\vec{x},\ \text{solution}}$$

3. Let S be the part of the cone $z^2 = x^2 + y^2$ between the planes $z = 1$ and $z = 4$ with upward unit normal vector $\vec{n}$ and let $\vec{F}(x, y, z)$ be the vector field

$$\vec{F}(x, y, z) = \langle 2x,\ z,\ y\rangle = 2x\vec{i} + z\vec{j} + y\vec{k}.$$

Evaluate the surface integral (the flux)

$$\iint_S \vec{F} \bullet \vec{n}\, dS$$

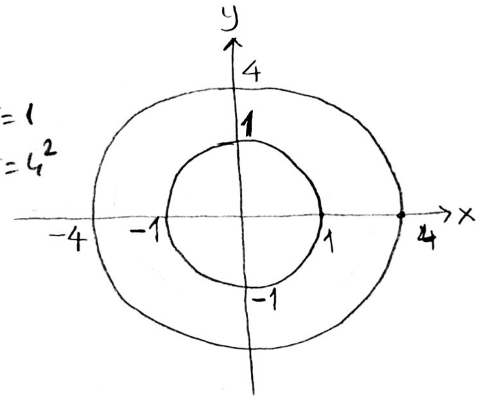
of $\vec{F}(x, y, z)$ over the surface S.

Solution:

$z^2 = x^2+y^2 \longrightarrow z = \sqrt{x^2+y^2}$

$z=1 \to x^2+y^2=1$

$z=4 \to x^2+y^2=4^2$



$x = r\cos\theta$

$y = r\sin\theta$ $\quad \left\{\begin{array}{l} 1 \le r \le 4 \\ 0 \le \theta \le 2\pi \end{array}\right\}$

$z = \sqrt{r^2} = r > 0$

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$

$\vec{r}_r = \langle \cos\theta, \sin\theta, 1\rangle$, $\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{N} = \vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\cos^2\theta + r\sin^2\theta\rangle = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

As $r>0$, $\vec{n} = \dfrac{\vec{N}}{\|\vec{N}\|}$, $\vec{n}\,dS = \dfrac{\vec{N}}{\|\vec{N}\|}\|\vec{N}\|\,dr\,d\theta = \langle -r\cos\theta, -r\sin\theta, r\rangle\,dr\,d\theta$

$$\iint_S \vec{F}\bullet\vec{n}\,dS = \int_0^{2\pi}\int_1^4 \langle 2r\cos\theta, r, r\sin\theta\rangle \circ \langle -r\cos\theta, -r\sin\theta, r\rangle\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^4 \left(-2r^2\cos^2\theta - r^2\sin\theta + r^2\sin\theta\right)dr\,d\theta = \int_0^{2\pi}\int_1^4 -r^2(1+\cos 2\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{r^3}{3}\Bigg|_1^4 (1+\cos 2\theta)\,d\theta = \int_0^{2\pi} -\left(\frac{64-1}{3}\right)(1+\cos 2\theta)\,d\theta = -21\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{2\pi}$$

$$= -21\left[2\pi + 0\right] = \boxed{-42\pi}$$

4. Let R be the solid region bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$, let the surface S be the boundary of the region R with outward unit normal vector $\vec{n}$, and let $\vec{F}(x,y,z)$ be the vector field

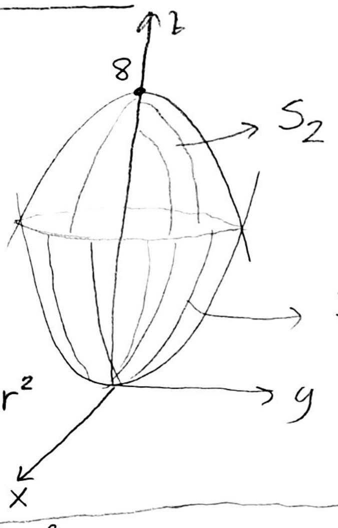
$$\vec{F}(x,y,z) = \left\langle e^{y^2 z} + 2x,\ y^2 + \cos(x^2 z),\ 4z \right\rangle = (e^{y^2 z} + 2x)\vec{\imath} + (y^2 + \cos(x^2 z))\vec{\jmath} + 4z\vec{k}.$$

Use the Divergence Theorem to evaluate the surface integral (the flux)

$$\iint_S \vec{F} \bullet \vec{n}\, dS$$

of $\vec{F}(x,y,z)$ over the surface S.

Solution:



S_1: part of $z = x^2+y^2$ is smooth

S_2: part of $z = 8-x^2-y^2$ is smooth,

so, $S = S_1 \cup S_2$ is piecewise smooth, closed, oriented surface. Thus by Divergence Theorem;

$r^2 \le z \le 8 - r^2$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_R \vec{\nabla} \cdot \vec{F}\, dV = \iiint_R (6+2y)\, dV$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}\left(e^{y^2 z} + 2x\right) + \frac{\partial}{\partial y}\left(y^2 + \cos(x^2 z)\right) + \frac{\partial}{\partial z}(4z)$$
$$= 2 + 2y + 4 = 6 + 2y$$

$$= 2\int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} (3 + r\sin\theta)\, dz\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^2 z\Big|_{r^2}^{8-r^2} (3 + r\sin\theta)\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^2 (8-2r^2)(3 + r\sin\theta)\, r\, dr\, d\theta$$

$$= 4\int_0^{2\pi}\int_0^2 \left(12r + 4r^2\sin\theta - 3r^3 - r^4\sin\theta\right) dr\, d\theta$$

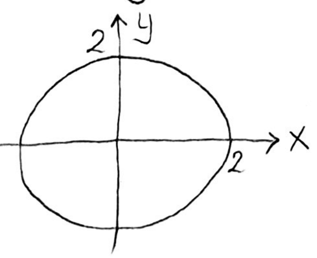
$$= 4\int_0^{2\pi} \left[6r^2 - \frac{3}{4}r^2 + \left(\frac{4}{3}r^3 - \frac{r^5}{5}\right)\sin\theta\right]\Bigg|_0^2 d\theta$$

$$= 4\int_0^{2\pi} \left(12 + \frac{64}{15}\sin\theta\right) d\theta = 4\left(12\theta - \frac{64}{15}\cos\theta\right)\Bigg|_0^{2\pi} = \boxed{96\pi}$$

Intersection curve:

$z = x^2 + y^2 = 8 - (x^2+y^2)$

$\Rightarrow x^2 + y^2 = 4$



$0 \le \theta \le 2\pi$

$0 \le r \le 2$