1) Let the reduced row echelon form of A be $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Determine A if the first, second and fourth columns of A are $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, respectively.

$$A = \begin{bmatrix} 1 & 0 & a & 1 \\ -1 & -1 & b & -2 \\ 3 & 1 & c & 0 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In order to find a, b, c values, you may apply row reduction directly.

$$\begin{bmatrix} 1 & 0 & a & 1 \\ -1 & -1 & b & -2 \\ 3 & 1 & c & 0 \end{bmatrix} \xrightarrow[-3R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & -1 & a+b & -1 \\ 0 & 1 & c-3a & -3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & c-3a & -3 \\ 0 & -1 & a+b & -1 \end{bmatrix} \xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & c-3a & -3 \\ 0 & 0 & b+c-2a & -4 \end{bmatrix}$$

$b + c - 2a = 0 \quad \Rightarrow \quad \boxed{b = 3}$

$c - 3a = -5 \quad \Rightarrow \quad \boxed{c = 1}$

$\boxed{a = 2}$

2) Solve the following linear system of equations.

$$x + y + z = 6$$
$$2x + 2y - z = 3$$
$$x - z = -2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 2 & -1 & 3 \\ 1 & 0 & -1 & -2 \end{array}\right] \xrightarrow[-2R_1+R_2 \to R_2]{-R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & -3 & -9 \\ 0 & -1 & -2 & -8 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{array}\right] \xrightarrow[-\frac{1}{3}R_3 \to R_3]{-R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x + y + z = 6$ --- (1)

$y + 2z = 8$ --- (2)

$\boxed{z = 3}$ --- (3)

Use (3) in eqn (2).

$y + 6 = 8 \Rightarrow \boxed{y = 2}$

Use (2) and (3) in eqn (1)

$x + 2 + 3 = 6$

$\Rightarrow \boxed{x = 1}$

There is a unique soln of this system.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

There is an isomorphism between $\mathbb{R}^4$ and P_3.
Therefore we can represent $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$ as vectors

3) Let W be a subspace of P_3. The set

$$S = \{\overbrace{-x - x^2 + x^3}^{\vec{v}_1}, \overbrace{1 + 2x + 2x^2 + 3x^3}^{\vec{v}_2}, \overbrace{2 + x + x^2 + 9x^3}^{\vec{v}_3}, \overbrace{1 - 2x - 2x^2 + 4x^3}^{\vec{v}_4}, \overbrace{-1 + 2x + 2x^2 - x^3}^{v_5}\}$$

spans W. Find a subset of S that is a basis for W.

$$\begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -2 & 2 \\ -1 & 2 & 1 & -2 & 2 \\ 1 & 3 & 9 & 4 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & 9 & 4 & -1 \\ -1 & 2 & 1 & -2 & 2 \\ -1 & 2 & 1 & -2 & 2 \\ 0 & 1 & 2 & 1 & -1 \end{bmatrix} \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2}$$

$$\begin{bmatrix} 1 & 3 & 9 & 4 & -1 \\ 0 & 5 & 10 & 2 & 1 \\ 0 & 5 & 10 & 2 & 1 \\ 0 & 1 & 2 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & 9 & 4 & -1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 5 & 10 & 2 & 1 \\ 0 & 5 & 10 & 2 & 1 \end{bmatrix} \xrightarrow[\substack{-5R_2+R_4 \to R_4 \\ -3R_2+R_1 \to R_1}]{-5R_2+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & -3 & 6 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_3 \to R_3} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 & 6 \end{bmatrix} \xrightarrow[\substack{-R_3+R_2 \to R_3 \\ 3R_3+R_4 \to R_4}]{-R_3+R_1 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's are on the 1st, the 2nd and the 4th columns.

Thus $\vec{v}_1, \vec{v}_2$ and $\vec{v}_4$ are linearly independent elts of this set,

$\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ is a basis for V.

1

4) Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 + 3x_3 - x_4 = 0\}$.
a. Show that V is a subspace of $\mathbb{R}^4$.
b. Find a basis for V.

a) let (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are elts of V

- $(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4)$

$$(x_1+y_1) - 2(x_2+y_2) + 3(x_3+y_3) - (x_4+y_4)$$
$$= (x_1 - 2x_2 + 3x_3 - x_4) + (y_1 - 2y_2 + 3y_3 - y_4) = 0 + 0 = 0$$

- $c(x_1, x_2, x_3, x_4) = (cx_1, cx_2, cx_3, cx_4)$

$$cx_1 - 2cx_2 + 3cx_3 - cx_4 = c(x_1 - 2x_2 + 3x_3 - x_4) = 0$$

Therefore it's a subspace.

b) $x_1 - 2x_2 + 3x_3 - x_4 = 0$

$$\begin{bmatrix} 1 & -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

The coefficient matrix is a row vector and the leading 1 is in the 1st column.

x_2, x_3, x_4 are free variables

$x_2 = p$, $x_3 = r$, $x_4 = s$

$x_1 = 2p - 3r + s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2p-3r+s \\ p \\ r \\ s \end{bmatrix} = p\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore a basis for V is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$