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1) Evaluate either $\iint\limits_S \vec{F}\cdot\vec{n}\,dS$ or $\iiint\limits_D \vec{\nabla}\cdot\vec{F}\,dV$ where

$$\vec{F} = x\,\vec{i} + y\,\vec{j},$$

D is the cylindrical region $1 \le x^2+y^2 \le 9$ between the planes $z=4$, $z=11$ and S is the surface of D.

$\vec{\nabla}\cdot\vec{F} = 1+1 = 2$

$1 \leq r \leq 3$

$$\iiint\limits_D \vec{\nabla}\cdot\vec{F}\,dV = 2\iiint\limits_D dV$$

$$= 2\int_0^{2\pi}\int_1^3\int_4^{11} dz\,r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_1^3 7r\,dr\,d\theta$$

$$= \int_0^{2\pi} 7r^2\Big|_1^3\,d\theta$$

$$= \int_0^{2\pi} 56\,d\theta$$

$$= 56\,\theta\Big|_0^{2\pi}$$

$$= 112\pi$$

2) a. Prove that $< f, g > = \int_0^\pi f(t)g(t)dt$ is an inner product.

- $\int_0^\pi f(t)\, g(t)\, dt = \int_0^\pi g(t)\, f(t) dt$
- $\int_0^\pi f(g+h) dt = \int_0^\pi fg\, dt + \int_0^\pi fh\, dt$
- $\int_0^\pi cfg\, dt = c\int_0^\pi fg\, dt$
- $\int_0^\pi f^2 dt \geq 0$ and

$\int_0^\pi f^2 dt = 0 \iff f$ is a zero func.

b. Apply Gram-Schmidt orthogonalization process to obtain an orthonormal basis for the vector space $V = span(S)$ where $S = \{\sin t, \cos t, 1\}$ and the inner product is defined as in part a.

u_1, u_2, u_3

$\sin t = u$

$\cos t dt = du$

$\sin^2 t = \frac{1 - \cos 2}{2}$

$\vec{V}_1 = \sin t$

$\vec{V}_2 = \cos t - \frac{\int_0^\pi \cos t \sin t\, dt}{\int_0^\pi \sin^2 t\, dt} \sin t$

$= \cos t - \frac{\frac{1}{2}\sin^2 t \,\Big|_0^\pi}{\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Big|_0^\pi} \sin t$ $) = 0$.

$= \cos t$.

$\vec{V}_3 = 1 - \frac{\int_0^\pi 1.\sin t\, dt}{\int_0^\pi \sin^2 t\, dt} \sin t - \frac{\int_0^\pi 1.\cos t\, dt}{\int_0^\pi \cos^2 t\, dt} \cos t$

$= 1 - \frac{4 \sin t}{\pi}$

$\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ is an orthogonal basis.

$\vec{w}_1 = \frac{\sin t}{\left(\int_0^\pi \sin^2 t\, dt\right)^{1/2}} = \frac{\sin t}{\sqrt{\pi/2}}$

$\vec{w}_2 = \frac{\cos t}{\left(\int_0^\pi \cos^2 t\, dt\right)^{1/2}} = \frac{\cos t}{\sqrt{\pi/2}}$

$\vec{w}_3 = \frac{1 - 4\sin t/\pi}{\left(\int_0^\pi \left(1 - \frac{4\sin t}{\pi}\right)^2 dt\right)^{1/2}} = \frac{1 - 4\sin t/\pi}{\sqrt{9\pi - 16}}$

$\{w_1, w_2, w_3\}$ is an orthonormal basis.

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3) For the matrix, $A = \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{bmatrix}$,

find an invertible matrix P and a digonal matrix D such that $D = P^{-1}AP$.

Check $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 7-\lambda & -4 & 0 \\ 8 & -5-\lambda & 0 \\ 6 & -6 & 3-\lambda \end{vmatrix} = (3-\lambda)\left[(\lambda-3)(\lambda+1)\right] = 0.$$

$\lambda = 3$ (double) and $\lambda = -1$ are eigenvalues.

For $\lambda = 3$ we have,

$$\begin{bmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1 \to R_1} \begin{bmatrix} 1 & -1 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix} \xrightarrow[-6R_1+R_3 \to R_3]{-8R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvectors for $\lambda = 3$ are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = -1$ we have,

$$\begin{bmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{bmatrix} \xrightarrow{\frac{1}{8}R_1 \to R_1} \begin{bmatrix} 1 & -1/2 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{bmatrix} \xrightarrow[-6R_1+R_3 \to R_3]{-8R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & -3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-1/3 R_2 \to R_2} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{1/2 R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvector for $\lambda = -1$ is $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

Thus a diagonal matrix D is $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

and the corresponding invertible matrix P is $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$.

4) Find a basis for the nullspace, rank and nullity of $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 0 & 9 & 0 \\ 0 & 4 & 8 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 0 & 9 & 0 \\ 0 & 4 & 8 & 0 \end{bmatrix} \xrightarrow[\frac{1}{4}R_3 \to R_3]{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

For a basis for nullspace, check $A\vec{x} = 0$.

$x_1 + 3x_3 = 0$ $\quad$ x_3, x_4 - free

$x_2 + 2x_3 = 0$ $\quad$ $x_3 = s$, $x_4 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3s \\ -2s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for nullspace is $\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

nullity = 2

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5) Find values of a, b, c and d so that $A = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \uparrow & \uparrow & \uparrow \end{array}$ $\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & b \\ 1/\sqrt{3} & 1/\sqrt{6} & c \\ a & -2/\sqrt{6} & d \end{bmatrix}$ is an orthogonal matrix.

$\|u_1\| = 1 \quad \frac{1}{3} + \frac{1}{3} + a^2 = 1 \Rightarrow a = \pm \frac{1}{\sqrt{3}}$

$\langle u_1, u_2 \rangle = 0 \quad \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{18}} - \frac{2a}{\sqrt{6}} = 0 \Rightarrow \boxed{a = \frac{1}{\sqrt{3}}}$

$\langle u_1, u_3 \rangle = 0 \quad \frac{b}{\sqrt{3}} + \frac{c}{\sqrt{3}} + \frac{d}{\sqrt{3}} = 0 \Rightarrow \boxed{b + c + d = 0}$

$\langle u_2, u_3 \rangle = 0 \quad \frac{b}{\sqrt{6}} + \frac{c}{\sqrt{6}} - \frac{2d}{\sqrt{6}} = 0 \Rightarrow \boxed{b + c - 2d = 0}$

If $b+c+d=0$ and $b+c-2d=0$ then $\boxed{d=0}$.
(and $\boxed{b+c=0}$)

Since A is orthogonal $AA^T = I = A^T A$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & b \\ 1/\sqrt{3} & 1/\sqrt{6} & c \\ 1/\sqrt{3} & -2/\sqrt{6} & d \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{3} + \frac{1}{6} + b^2 = 1 \Rightarrow b = \pm 1/\sqrt{2}$.

$\frac{1}{3} + \frac{1}{6} + c^2 = 1 \Rightarrow c = \pm 1/\sqrt{2}$.

Since $b + c = 0$,

if $b = 1/\sqrt{2}$ then $c = -1/\sqrt{2}$

if $b = -1/\sqrt{2}$ then $c = 1/\sqrt{2}$