1) Find the volume of the solid above the cone $z = \sqrt{x^2+y^2}$ and below the paraboloid $z = 12 - x^2 - y^2$.

$z = r$

$z = 12 - r^2$

$12 - r^2 = r$

$r^2 + r - 12 = 0$

$(r+4)(r-3) = 0 \Rightarrow r = 3$ (radius of the disk of the intersection)

Use cylindrical coordinates.

$$\int_0^{2\pi}\int_0^{3}\int_r^{12-r^2} dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{3} (12 - r^2 - r)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{3} (12r - r^3 - r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(6r^2 - \frac{r^4}{4} - \frac{r^3}{3} \Big|_0^3 \right) d\theta$$

$$= \int_0^{2\pi} \left(54 - \frac{81}{4} - 9 \right) d\theta$$

$$= \frac{99}{4} \int_0^{2\pi} d\theta$$

$$= \frac{99}{4} \cdot 2\pi$$

$$= \frac{99\pi}{2}$$

2) Evaluate the line integral $\int_C f(x,y)\,ds$ where $f = \frac{y}{x}$ and C is the line segment from $(1,4)$ to $(3,10)$.

Parametrize the line.

$x(t) = 1+2t \qquad 0 \leq t \leq 1$

$y(t) = 4+6t$

$ds = \sqrt{4+36}\,dt$

$= 2\sqrt{10}\,dt$

$f(t) = \frac{4+6t}{1+2t}$

$\int_0^1 \frac{4+6t}{1+2t}\, 2\sqrt{10}\,dt$

$= 2\sqrt{10} \int_0^1 \frac{1+3(1+2t)}{1+2t}\,dt$

$= 2\sqrt{10}\left(\int_0^1 \frac{1}{1+2t}\,dt + \int_0^1 3\,dt\right)$

$= 2\sqrt{10}.3 + 2\sqrt{10}\int_0^1 \frac{1}{1+2t}\,dt \qquad u = 1+2t, \quad du = 2dt \text{ or } \frac{du}{2} = \text{(}$

$= 6\sqrt{10} + 2\sqrt{10}\int_1^3 \frac{1}{2u}\,du$

$= 6\sqrt{10} + 2\sqrt{10}\,\frac{1}{2}\ln u \Big|_1^3$

$= 6\sqrt{10} + \sqrt{10}\,(\ln 3 - \overbrace{\ln 1}^{=0})$

$= 6\sqrt{10} + \sqrt{10}\ln 3$

3) Verify thet the following vector field is conservative. Find a potential function f.

$$\overrightarrow{F} = \underbrace{(6xy^2 + 5x^4)}_{P}\,\vec{\imath} + \underbrace{(4y^3 + 6x^2y)}_{Q}\,\vec{\jmath}$$

$P_y = 12xy = Q_x$

Therefore, $\vec{F}$ is conservative. ($P = f_x$ and $Q = f_y$)

$f = \int (6xy^2 + 5x^4)\,dx$

$= 3x^2y^2 + x^5 + h(y)$ --- (1)

Since $Q = f_y$, take der. of (1) with respect to y and the result must be equal to Q.

$\frac{\partial}{\partial y}(3x^2y^2 + x^5 + h(y)) = 6x^2y + h'(y) = 4y^3 + 6x^2y$

we conclude that $h'(y) = 4y^3$

To determine $h(y)$ take integral with respect to y

$\int 4y^3\,dy = y^4 + C$

we obtain, $f(x,y) = 3x^2y^2 + x^5 + y^4 + C$.

4) Evaluate $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $\overrightarrow{F} = (2y + \sqrt{1+x^5})\,\vec{\imath} + (5x - e^{y^2})\,\vec{\jmath}$ and C is the circle $x^2 + y^2 = 4$.

$\oint_C$ refers that the conditions of Green's thm is satisfied Therefore we may use $\iint_A (Q_x - P_y)\,dA$ instead of $\oint_C \vec{F} \cdot d\vec{r}$

$Q_x - P_y = 5 - 2 = 3$

$\iint_A 3\,dA = 3\iint_A dA$ (area is a disk with radius 2)

$= 3\pi 4$

$= 12\pi$

5) Find the surface area that is a part of the plane $3x + 4y + 10z = 0$ inside the cylinder $x^2 + y^2 = 1$.

$$z = \frac{-3x - 4y}{10} = f(x,y)$$

Calculate $\sqrt{f_x^2 + f_y^2 + 1}$.

$$\sqrt{\left(-\frac{3}{10}\right)^2 + \left(\frac{-4}{10}\right)^2 + 1} = \sqrt{\frac{125}{100}} = \frac{5\sqrt{5}}{10} = \frac{\sqrt{5}}{2}$$

$$\iint_S dS = \iint_A \frac{\sqrt{5}}{2}\, dA$$

Projection on xy-plane is a disk with radius 1. And since it's a disk we should use polar coordinates

$$\int_0^{2\pi}\int_0^1 \frac{\sqrt{5}}{2}\, r\,dr\,d\theta$$

$$= \frac{\sqrt{5}}{2}\pi r^2 \text{ (where } r = 1)$$

$$= \frac{\sqrt{5}\,\pi}{2}$$