ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 255 - Vector Calculus and Linear Algebra

FIRST MIDTERM EXAMINATION

14.03.2017

SAMPLE SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 90 minutes

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

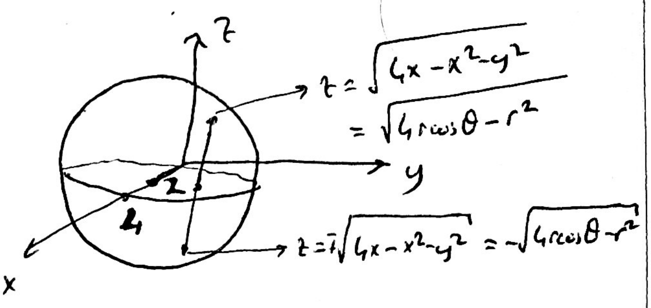
2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1) Let D be the solid region bounded by the sphere $x^2+y^2+z^2=4x$. Write $\int\int\int_D f(x,y,z)dV$ as an iterated triple integral in

(a) Cylindrical Coordinates,

(b) Spherical Coordinates.

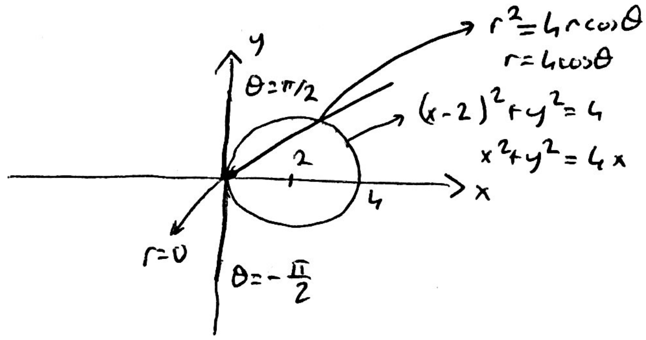


$x^2+y^2+z^2=4x$

$x^2-4x+4+y^2+z^2=4$

$(x-2)^2+y^2+z^2=4.$

On xy-plane $(x-2)^2+y^2=4$



a) $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$

$0\leq r\leq 4\cos\theta$

$-\sqrt{4r\cos\theta-r^2}\leq z\leq\sqrt{4r\cos\theta-r^2}$

$$\iiint_D f(x,y,z)\,dV=\int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta}\int_{-\sqrt{4r\cos\theta-r^2}}^{\sqrt{4r\cos\theta-r^2}} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

b) $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$

$0\leq\phi\leq\pi$

$0\leq\rho\leq 4\sin\phi\cos\theta$

$x^2+y^2+z^2=4x$

$\rho^2=4\rho\sin\phi\cos\theta$

$\rho=4\sin\phi\cos\theta$

$$\iiint_D f(x,y,z)\,dV=\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{4\sin\phi\cos\theta} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

2) Let C be the curve parametrized by

$$\vec{r}(t) = t\vec{i} + t^2\vec{j} + \frac{4}{3}t^{3/2}\vec{k}$$

with $1 \le t \le 4$. Evaluate the line integral $\displaystyle\int_C \frac{15z - 3y}{2x+1}\,ds$.

$$ds = \left|\frac{d\vec{r}}{dt}\right| dt$$

$$= \left| 1\vec{i} + 2t\,\vec{j} + \frac{4}{3}\cdot\frac{3}{2}\,t^{1/2}\,\vec{k} \right| dt$$

$$= \sqrt{1^2 + (2t)^2 + (2t^{1/2})^2}\; dt$$

$$= \sqrt{4t^2 + 4t + 1}\; dt$$

$$= (2t+1)\,dt \qquad \text{as } t > \frac{-1}{2}$$

$$\int_C \frac{15z - 3y}{2x+1}\,ds = \int_1^4 \frac{\cancel{15}^{5} \cdot \frac{4}{\cancel{3}}\,t^{3/2} - 3t^2}{\cancel{2t+1}}\,\cancel{(2t+1)}\,dt$$

$$= \int_1^4 (20\,t^{3/2} - 3t^2)\,dt$$

$$= \left(20 \cdot \frac{t^{5/2}}{\frac{5}{2}} - t^3\right)\Bigg|_1^4$$

$$= \left(\cancel{20}^{4} \cdot \frac{2}{\cancel{5}}\,t^{5/2} - t^3\right)\Bigg|_1^4$$

$$= 8 \cdot 4^{5/2} - 4^3 - (8 \cdot 1^{5/2} - 1^3)$$

$$= 8 \cdot 32 - 64 - 8 + 1$$

$$= 185$$

3) Let C be the line of the intersection of the planes $x+y-z=2$ and $2x-y+2z=1$ from $(1,1,0)$ to $(0,5,3)$ and

$$\vec{F}(x,y,z)=e^{y}\vec{\imath}+\cos z\vec{\jmath}+x^2\vec{k}$$

Evaluate the line integral $\int_C \vec{F}\bullet d\vec{r}$.

Since C is the line segment joining $(1,1,0)$ to $(0,5,3)$

$$\vec{r}(t)=(1,1,0)+t\big((0,5,3)-(1,1,0)\big)$$

$$=(1-t,\,1+4t,\,3t) \qquad 0\leq t\leq 1$$ is a parametrization of C.

$$\begin{aligned} x(t)&=1-t & & dx=-dt \\ y(t)&=1+4t & \longrightarrow\quad & dy=4dt & \qquad d\vec{r}=\langle -1,4,3\rangle dt \\ z(t)&=3t & & dz=3dt \end{aligned}$$

$$F(\vec{r}(t))=\langle e^{1+4t},\cos(3t),(1-t)^2\rangle$$

$$\int_C \vec{F}\cdot d\vec{r}=\int_0^1 \langle e^{1+4t},\cos(3t),(1-t)^2\rangle\cdot\langle -1,4,3\rangle dt$$

$$=\int_0^1\left(-e^{1+4t}+4\cos(3t)+3(1-t)^2\right)dt$$

$$=\left(-\frac{e^{1+4t}}{4}+\frac{4\sin(3t)}{3}+\frac{3(1-t)^3}{-3}\right)\Big|_0^1$$

$$=-\frac{e^5}{4}+\frac{4}{3}\sin 3-(1-1)^3-\left[-\frac{e^1}{4}+\frac{4\sin 0}{3}-(1-0)^3\right]$$

$$=-\frac{e^5}{4}+\frac{4}{3}\sin 3+\frac{e}{4}+1$$

4) Let C be the curve with parametrization

$$\vec{r}(t) = \cos(\pi t^2)\vec{i} + \ln(e^t - t + \overset{2}{\cancel{1}})\vec{j} + t^{2017}\vec{k}$$

with $0 \le t \le 1$ and

$$\vec{F}(x,y,z) = \left(\frac{1}{1+x^2} - 2xy + yz\cos(xyz)\right)\vec{i} + \left(xz\cos(xyz) - x^2\right)\vec{j} + \left(\frac{1}{1+z} + xy\cos(xyz)\right)\vec{k}$$

Evaluate the line integral $\displaystyle\int_C \vec{F} \bullet d\vec{r}$.

Let $M(x,y,z) = \frac{1}{1+x^2} - 2xy + yz\cos(xyz)$

$N(x,y,z) = xz\cos(xyz) - x^2$

$P(x,y,z) = \frac{1}{1+z} + xy\cos(xyz)$

$$\left.\begin{aligned} M_y &= -2x + z\cos(xyz) - xyz^2\sin(xyz) \\ N_x &= z\cos(xyz) - xyz^2\sin(xyz) - 2x \end{aligned}\right\} \quad M_y = N_x$$

$$\left.\begin{aligned} M_z &= y\cos(xyz) - xy^2z\sin(xyz) \\ P_x &= y\cos(xyz) - xy^2z\sin(xyz) \end{aligned}\right\} \quad M_z = P_x$$

$$\left.\begin{aligned} N_z &= x\cos(xyz) - x^2yz\sin(xyz) \\ P_y &= x\cos(xyz) - x^2yz\sin(xyz) \end{aligned}\right\} \quad N_z = P_y$$

Thus, $\vec{F}$ is conservative.

Let $\phi(x,y,z)$ be a potential function for $F(x,y,z) = \langle M, N, P\rangle$. Then

$\phi_x(x,y,z) = M(x,y,z) = \frac{1}{1+x^2} - 2xy + yz\cos(xyz)$

$\phi_y(x,y,z) = N(x,y,z) = xz\cos(xyz) - x^2$

$\phi_z(x,y,z) = P(x,y,z) = \frac{1}{1+z} + xy\cos(xyz)$

Now, $\phi(x,y,z) = \int \phi_x(x,y,z)\,dx = \int\left(\frac{1}{1+x^2} - 2xy + yz\cos(xyz)\right)dx$

$= \arctan x - x^2y + \sin(xyz) + g(y,z).$

$xz\cos(xyz) - x^2 = \phi_y(x,y,z) = -x^2 + xz\cos xyz + g_y(y,z) \longrightarrow g_y(y,z) = 0.$

So $g(y,z) = \int 0\,dy = h(z).$

Hence $\phi(x,y,z) = \arctan x - x^2y + \sin(xyz) + h(z).$

$$\frac{1}{1+z} + xy\cos(xyz) = \phi_z(x,y,z) = xy\cos(xyz) + h'(z)$$

$$h'(z) = \frac{1}{1+z}$$

$$h(z) = \int \frac{1}{1+z}\,dz = \ln|1+z| + c.$$

So $\phi(x,y,z) = \arctan x - x^2y + \sin(xyz) + \ln|1+z| + c$ is a potential for $\vec{F}(x,y,z)$

$$\vec{r}(0) = \cos 0\ \vec{i} + \ln(e^0 - 0 + 0^2)\vec{j} + 0\vec{k}$$

$$= (1,0,0)$$

$$\vec{r}(1) = \cos\pi\ \vec{i} + \ln(e^1 - 1 + 1^2)\vec{j} + 1^{2017}\vec{k}$$

$$= (-1, 1, 1)$$

By the Fundamental Theorem of Line Integrals

$$\int_C \vec{F}\cdot d\vec{r} = \phi(\vec{r}(1)) - \phi(\vec{r}(0))$$

$$= \phi(-1,1,1) - \phi(1,0,0)$$

$$= \arctan(-1) - (-1)^2\cdot 1 + \sin(-1\cdot 1\cdot 1) + \ln|1+1| + c$$

$$- \left(\arctan 1 - 1^2\cdot 0 + \sin(1\cdot 0\cdot 0) + \ln|0+1| + c\right)$$

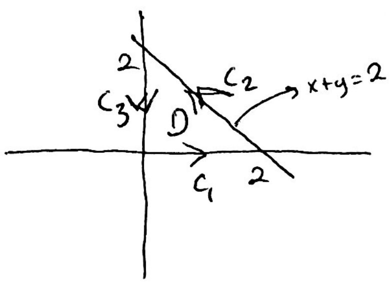
$$= -\frac{\pi}{4} - 1 + \sin(-1) + \ln 2 + c - \frac{\pi}{4} - c$$

$$= -\frac{\pi}{2} - 1 - \sin 1 + \ln 2$$

5) Let D be the region in the upper half plane bounded by the x-axis, y-axis and the line $x + y = 2$. Evaluate the line integral

$$\oint_C \left[2xy - xy^2 + \ln(1 + x^{2016})\right] dx + \left[x^2y + 2x - \sin(y^3)\right] dy$$

where C is the positively oriented boundary of the region D.



C_1, C_2, C_3 are line segments. So they are smooth. Hence $C = C_1 + C_2 + C_3$ is piecewise smooth.

C is simple closed.

C is positively oriented boundary of D.

$M(x,y) = 2xy - xy^2 + \ln(1 + x^{2016})$

$N(x,y) = x^2y + 2x - \sin(y^3)$

$M_x = 2y - y^2 + \dfrac{x^{2015}}{1+x^{2016}}$

$M_y = 2x - 2xy$

$N_x = 2xy + 2$

$N_y = x^2 - 3y^2\cos(y^3)$ are all continuous on $\mathbb{R}^2$.

Then by Green's Theorem

$$\oint_C M\,dx + N\,dy = \iint_D (N_x - M_y)\, dA$$

$0 \leq x \leq 2$

$0 \leq y \leq 2-x$

$$= \iint_D (2xy + 2 - 2x + 2xy)\, dA$$

$$= \int_0^2 \int_0^{2-x} (4xy + 2 - 2x)\, dy\, dx$$

$$= \int_0^2 \left(2xy^2 + 2y - 2xy\right)\Big|_0^{2-x} dx$$

$$= \int_0^2 \left(2x(4 - 4x + x^2) + 4 - 2x - 4x + 2x^2\right) dx$$

$$= \int_0^2 (2x - 6x^2 + 2x^3 + 4)\, dx = x^2 - 2x^3 + \frac{2}{4}x^4 + 4x\Big|_0^2 = 4 - 16 + \frac{1}{2}\cdot 16 + 4\cdot 2$$

$$= 4.$$